Twelve Cardinals and their Relations

The consonances are those intervals which are formed from the natural steps.
An interval may be diminished when one of its steps is replaced by a smaller one.
Or it may be augmented when one of its steps is replaced by a larger one.

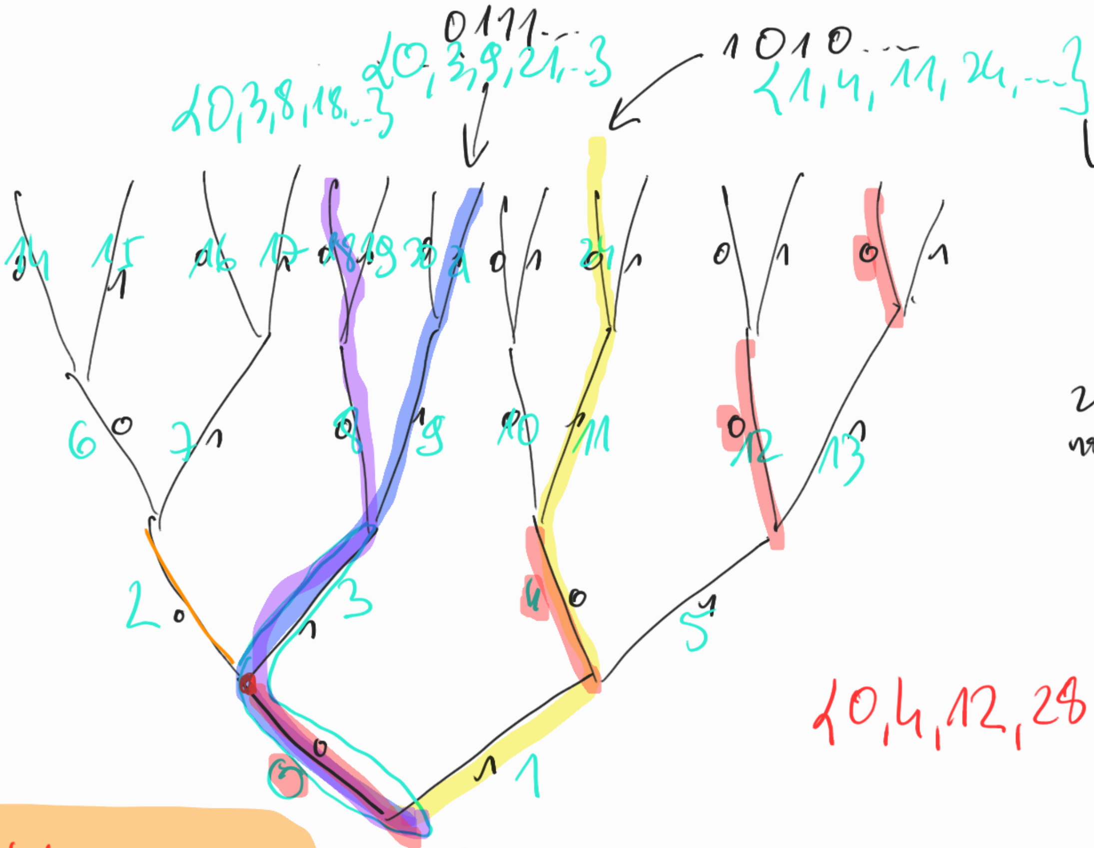
GIOSEFFO ZARLINO
Le Istitutioni Harmoniche, 1558

In this chapter we investigate twelve cardinal characteristics and their relations to one another. A cardinal characteristic of the continuum is an uncountable cardinal number which is less than or equal to $\mathfrak{c}$ that describes a combinatorial or analytical property of the continuum. Like the power of the continuum itself, the size of a cardinal characteristic is often independent from ZFC. However, some restrictions on possible sizes follow from ZFC, and we shall give a complete list of what is known to be provable in ZFC about their relation. Later in Part II, but mainly in Part III, we shall see how one can diminish or augment some of these twelve cardinals without changing certain other cardinals. In fact, these cardinal characteristics are also used to investigate combinatorial properties of the various forcing notions introduced in Part III.

forsing

We shall encounter some of these cardinal characteristics (*e.g.*, $\mathfrak{p}$) more often than others (*e.g.*, $\mathfrak{i}$). However, we shall encounter each of these twelve cardinals again, and like the twelve notes of the chromatic scale, these twelve cardinals will build the framework of our investigation of the combinatorial properties of forcing notions that is carried out in Part III.

On the one hand, it would be good to have the definition of a cardinal characteristic at hand when it is needed; but on the other hand, it is also convenient to have all the definitions together (especially when a cardinal characteristic is used several times), rather than scattered over the entire



$|\text{gałęzi}| = |\{0,1\}^{\mathbb{N}}| = |\mathbb{R}| = \mathfrak{c}$

zbiór wszystkich nsk ciągów 0 i 1.

Kolekcja: prawie rozłączna. – zbiory wyznaczone przez wszystkie gałęzie mocy $\mathfrak{c}$

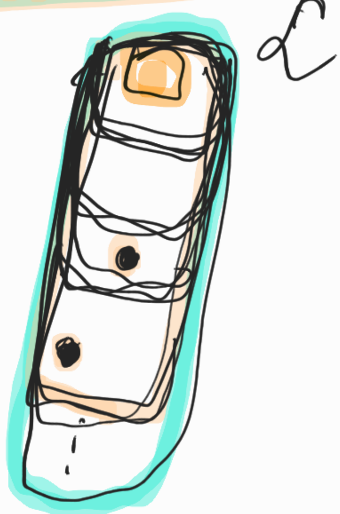
nie jest maksymalna

Tw. Jeśli $\mathcal{A}$ jest rodziną zbiorów prawie rozłącznych to istnieje rodzina zbiorów prawie rozłącznych $\mathcal{B}$, która jest maksymalna i $\mathcal{A} \subseteq \mathcal{B}$.

Lemat K-Z. $X, \leq$ – częściowy porządek $X \neq \emptyset$ oraz każdy łańcuch w X ma ograniczenie górne to w X istnieje element maksymalny.

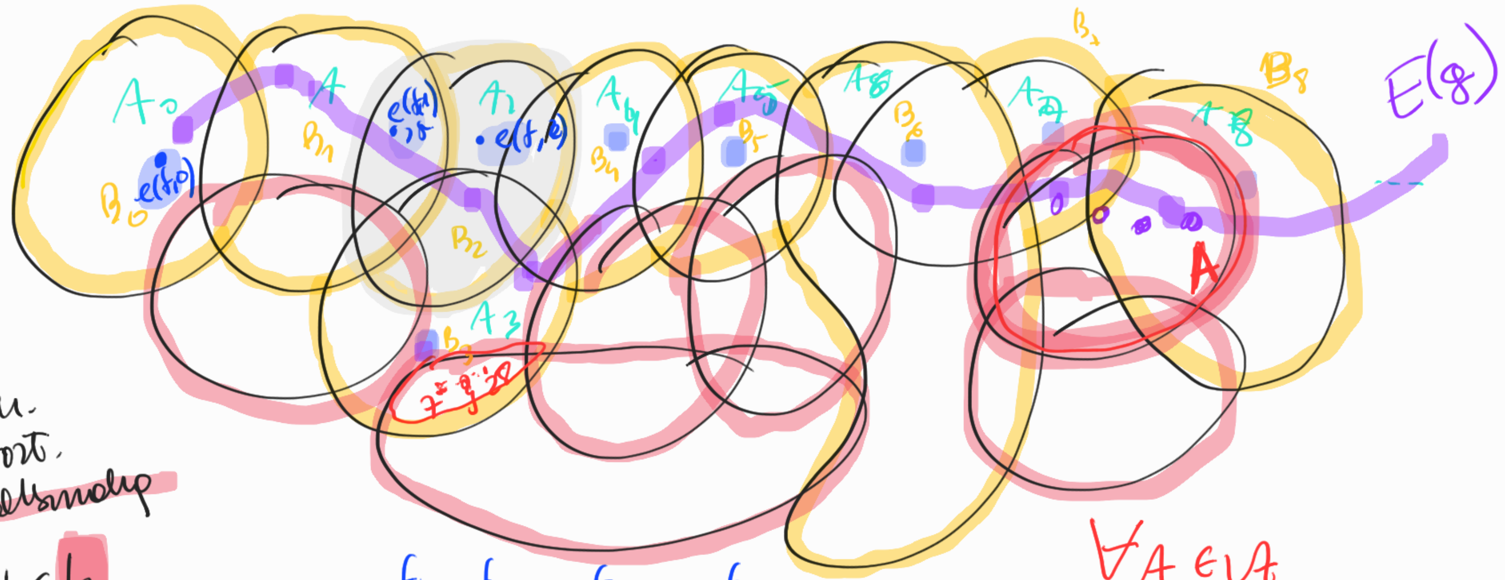
D. X wszystkie rodziny zbiorów prawie rozłącznych które zawierają rodzinę $\mathcal{A}$

$, \subseteq$



Wniosek. Istnieje maksymalna rodzina prawie rozłączna mocy $\mathfrak{c}$

$\mathcal{A}$



rodz. p. rozł.
~~[illegible]~~

$|\mathcal{A}| < \mathfrak{b}$

$f: \overset{f_0}{1} \quad \overset{f_1}{5} \quad \overset{f_2}{8} \quad \overset{f_3}{2020} \quad \ldots$

$\forall A \in \mathcal{A}$ niepoliczonego

$f(A): \underbrace{\quad}_{\supset A \cap A_0} \quad \underbrace{\quad}_{\supset A \cap A_1} \quad \underbrace{\quad}_{\supset A \cap A_2} \quad 29,$

$e(f,n)$ min element $B_n > f_n$

$\forall_f \quad E(f) = \{e(f,0), e(f,1), \ldots\} \quad \forall_n \quad E(f)$ p.r z A_n

$A \not\subseteq$

od pewnego n

$\boxed{f(A)_n} < g_n < e(g,n)$

$\cap \quad \cap$

$B_n \quad A_n$

$\forall_A \quad f(A) <^* g$

$E(g) = \{\underline{e(g,0), e(p,1)}, \boxed{e(g,2)}, \ldots\}$

$\uparrow$

min B_2

$> g_2$

liczba $>$

~~od~~ dowolnej liczby

w $\boxed{A_n \cap A}$

book. Defining all twelve cardinals at once also gives us the opportunity to show what is known to be provable in ZFC about the relationship between these twelve cardinals. Thus, one might first skip this chapter and go back to it later and take bits and pieces when necessary.

$\aleph_1$

miesz. ciągi 0-1 $\{0,1\}^{\mathbb{N}}$ $\mathbb{N}^{\mathbb{N}}$ ← wsz. ciągi l. naturalnych

The Cardinals ω_1 and $\mathfrak{c}$

$\mathbb{N} = \omega = \aleph_0$

We have already met both cardinals, $\mathfrak{c}$ and ω_1: $\mathfrak{c}$ is the cardinality of the continuum $\mathbb{R}$, and ω_1 is the smallest uncountable cardinal. According to FACT 4.3, $\mathfrak{c} = 2^\omega$ is also the cardinality of the sets $[0,1]$, ${}^\omega 2$, ${}^\omega\omega$, and $[0,1] \setminus \mathbb{Q}$; and by LEMMA 4.10, ω_1 can also be considered as the set of order types of well-orderings of $\mathbb{Q}$.

uogólniona HC

$|P(A)| > |A|$

$|P(\aleph_\alpha)| = \aleph_{\alpha+1}$

The Continuum Hypothesis, denoted CH, states that $\mathfrak{c}$ is the least uncountable cardinal, *i.e.*, $\mathfrak{c} = \omega_1$ (*cf.* Chapter 4), which is equivalent to saying that every subset of $\mathbb{R}$ is either countable or of the same cardinality as $\mathbb{R}$. Furthermore, the Generalised Continuum Hypothesis, denoted GCH, states that for every ordinal $\alpha \in \Omega$, $2^{\omega_\alpha} = \omega_{\alpha+1}$. Gödel showed that $\mathbf{L} \vDash$ GCH, where $\mathbf{L}$ is the constructible universe (see the corresponding note in Chapter 5), thus, GCH is consistent with ZFC.

$> \aleph_0$ $\geq \aleph_1$ $\leq$ ← niezmienniki kardynalne

$|\mathbb{R}| = \mathfrak{c}$

$\geq$

κ

$\geq$

$\aleph_1$

$\geq$

$|\mathbb{N}| = \aleph_0$

(CH) $\aleph_1 = \mathfrak{c}$

Each of the following ten combinatorial cardinal characteristics of the continuum is uncountable and less than or equal to $\mathfrak{c}$. Thus, if we assume CH, then these cardinals are all equal to $\mathfrak{c}$. However, as we shall see in Part II, CH is not provable in ZFC. In other words, if ZFC is consistent then there are models of ZFC in which CH fails, *i.e.*, models in which $\omega_1 < \mathfrak{c}$. In those models, possible (*i.e.*, consistent) relations between the following cardinal characteristics will be provided in Part II and Part III.

$\aleph_1 \neq \mathfrak{c}$

The Cardinal $\mathfrak{p}$

IN ko-skończony prawie zawarty

For two sets $x, y \subseteq \omega$ we say that x is **almost contained** in y, denoted $x \subseteq^* y$, if $x \setminus y$ is finite, *i.e.*, all but finitely many elements of x belong to y. For example a finite subset of ω is almost contained in $\emptyset$, and ω is almost contained in every co-finite subset of ω (*i.e.*, in every $y \subseteq \omega$ such that $\omega \setminus y$ is finite). A **pseudo-intersection** of a family $\mathscr{F} \subseteq [\omega]^\omega$ of infinite subsets of ω is an infinite subset of ω that is almost contained in every member of $\mathscr{F}$. For example ω is a pseudo-intersection of the family of co-finite sets. Furthermore, a family $\mathscr{F} \subseteq [\omega]^\omega$ has the **strong finite intersection property** (*sfip*) if every finite subfamily has infinite intersection. Notice that every family with a pseudo-intersection necessarily has the *sfip*, but not vice versa. For example any filter $\mathscr{F} \subseteq [\omega]^\omega$ has the *sfip*, but no ultrafilter on $[\omega]^\omega$ has a pseudo-intersection.

nie psp ⇒ nie sfip

⇍

ultrafiltry

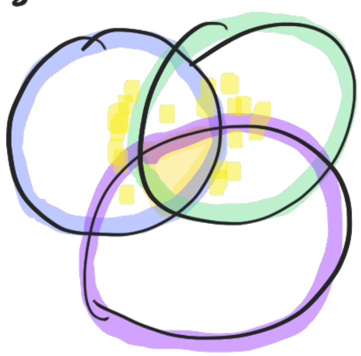
pseudo przecięcie

silna własność skończonego przecięcia

nie

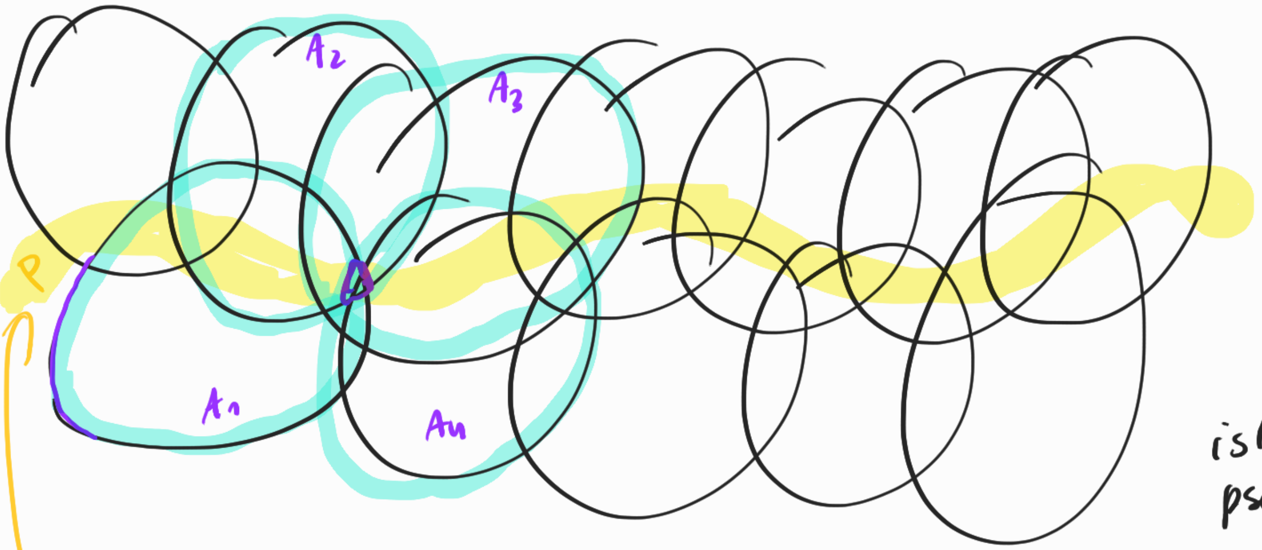
$[A]^k$ - zbiór wszystkich k-elementowych podzbiorów zbioru A.

$[\omega]^\omega$ - zbiór wszystkich nieskończonych podzbiorów zb. IN



przecięcie - zawarte w każdym z przecinanych zbiorów

pseudo-przecięcie - prawie zawarte w każdym z przecinanych zbiorów



...

nie sk. rodzina niesk. zbiorów

istnieje pseudoprzecięcie

1° jest nsk.

2° prawie zawarty w każdym z tych zbiorów

T: ta rodzina ma sfip.

$\Leftrightarrow$

skończenie wiele zbiorów z mojej rodziny ma nsk przecięcie

$P \setminus A_1$ skończony $\qquad$ $P \cap A_1$ nieskończony

$(P\cap A_1)\setminus A_2$ skończony $\qquad$ $P\cap A_1 \cap A_2$ nieskończony

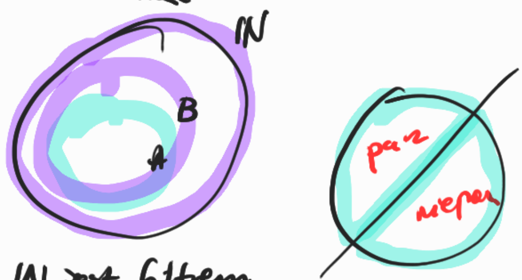
...

$P \cap \boxed{A_1\cap A_2 \cap A_3 \cap A_n}$ nieskończony

ultrafiltr

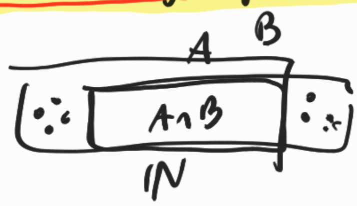
Filtr $\mathcal{F}$ to rodzina podzbiorów $\mathbb{N}$ spełniająca nast. warunki:

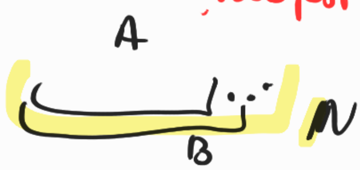
- $\emptyset \notin \mathcal{F}$ $\qquad$ $\mathbb{N} \in \mathcal{F}$
- $A \in \mathcal{F}$ $\qquad$ $A \subseteq B \Rightarrow B \in \mathcal{F}$
- $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
- $A \cup B = \mathbb{N} \Rightarrow A \in \mathcal{F} \vee B \in \mathcal{F}$



Rodzina wszystkich ko-skończonych podzb. zbioru $\mathbb{N}$ jest filtrem, ale nie jest ultrafiltrem

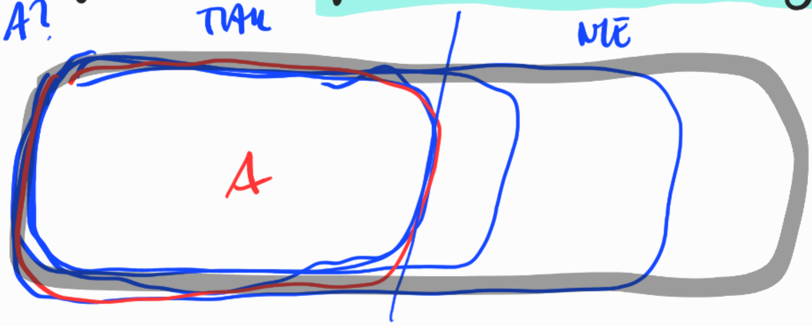
przerwa 18:00





Przykład. $\{A \subseteq \mathbb{N} : 2021 \in A\}$ - ultrafiltr główne

czy A? TAK NIE



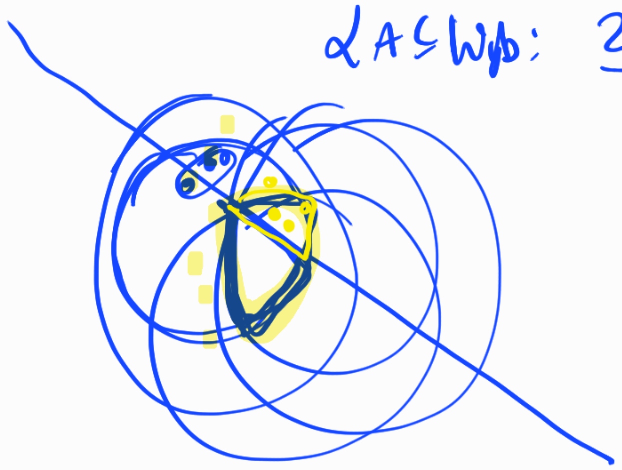
$\mathcal{F}$

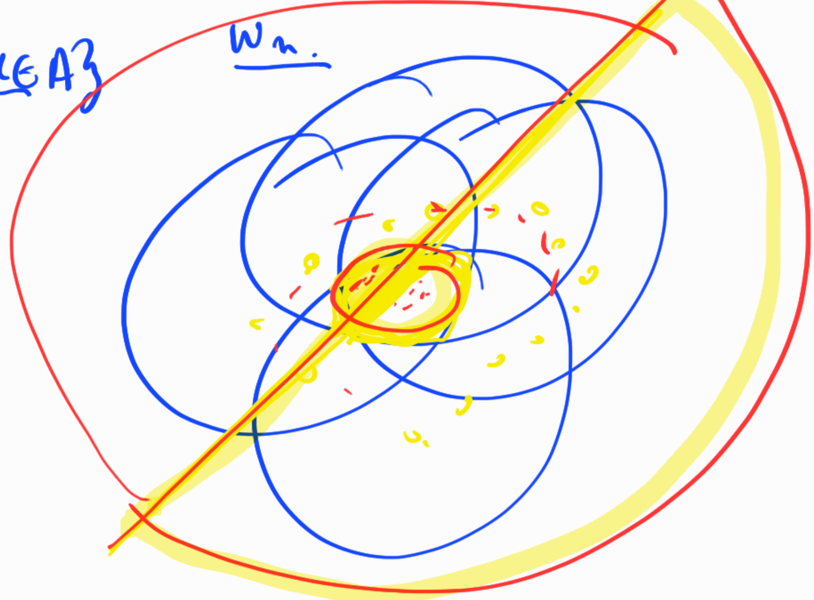
Tw. Arrow '70

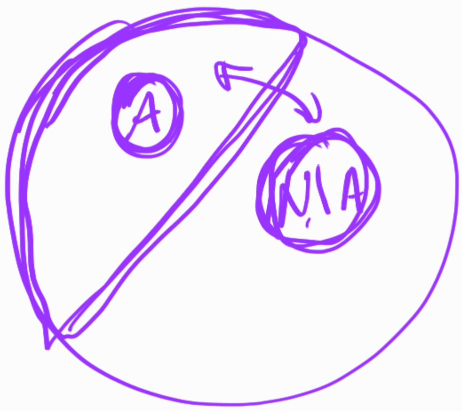
- jest opisany ultrafiltrem

$\{A \subseteq W_{fb} : \text{zero} \in A\}$

W_n.







$|\mathcal{P}(\mathbb{N})| = \mathfrak{c}$

Tw. $\aleph_0^+ < \mathfrak{c}$

T: Przeliczalna rodzina zbiorów nie jest rozdzielająca

$= \{X_0, X_1, \ldots\}$

$= \{x_0, \ldots\}$

$\exists\, Y \subseteq \mathbb{N}$ nie jest rozdzielany przez żadnego X-a

y_0	y_1	y_2	y_3
ogran. $\in X_0$	ogr $\cap X_1$ nsk ogr $\setminus X_1$ nsk	ogr $\cap X_2$ nsk ogr $\setminus X_2$ nsk	ogr $\cap X_3$ nsk ogr $\setminus X_3$ nsk
	$\in X_1$	$\in X_2$	$\in X_3$

. . .

DEFINITION OF $\mathfrak{p}$. The **pseudo-intersection number** $\mathfrak{p}$ is the smallest cardinality of any family $\mathscr{F} \subseteq [\omega]^\omega$ which has the *sfip* but which does not have a pseudo-intersection; more formally

$$\mathfrak{p} = \min\big\{|\mathscr{F}| : \mathscr{F} \subseteq [\omega]^\omega \text{ has the sfip but no pseudo-intersection}\big\}.$$

Since ultrafilters on $[\omega]^\omega$ are families which have the *sfip* but do not have a pseudo-intersection, and since every ultrafilter on $[\omega]^\omega$ is of cardinality $\mathfrak{c}$, the cardinal $\mathfrak{p}$ is well-defined and $\mathfrak{p} \le \mathfrak{c}$. It is natural to ask whether $\mathfrak{p}$ can be smaller than $\mathfrak{c}$; however, the following result shows that $\mathfrak{p}$ cannot be too small.

$\aleph_0 = \mathbb{N}$ $\quad$ $\aleph_0 < \mathfrak{p}$

THEOREM 8.1. $\omega_1 \le \mathfrak{p}$.

Proof. Let $\mathscr{E} = \{X_n \in [\omega]^\omega : n \in \omega\}$ be a countable family which has the *sfip*. We construct a pseudo-intersection of $\mathscr{E}$ as follows: Let $a_0 := \bigcap X_0$ and for positive integers n let

$$a_n = \bigcap\Big(\bigcap\{X_i : i \in n\} \setminus \{a_i : i \in n\}\Big).$$

Further, let $Y = \{a_n : n \in \omega\}$; then for every $n \in \omega$, $Y \setminus \{a_i : i \in n\} \subseteq X_n$ which shows that $Y \subseteq^* X_n$, hence, Y is a pseudo-intersection of $\mathscr{E}$. ⊣

$a_2 = \min\big(X_0 \cap X_1 \setminus \{a_0, a_1\}\big)$

$a_3 = \min\big(X_0 \cap X_1 \cap X_2 \setminus \{a_0, a_1, a_2\}\big)$

19:00 prerwa

The Cardinals $\mathfrak{b}$ and $\mathfrak{d}$

For two functions $f, g \in {}^\omega\omega$ we say that g **dominates** f, denoted $f <^* g$, if for all but finitely many integers $k \in \omega$, $f(k) < g(k)$, *i.e.*, if there is an $n_0 \in \omega$ such that for all $k \ge n_0$, $f(k) < g(k)$. Notice that ordering "$<^*$" is transitive, however, "$<^*$" it is not a linear ordering (we leave it as an exercise to the reader to find functions $f, g \in {}^\omega\omega$ such that neither $f <^* g$ nor $g <^* f$).

A family $\mathscr{D} \subseteq {}^\omega\omega$ is **dominating** if for each $f \in {}^\omega\omega$ there is a function $g \in \mathscr{D}$ such that $f <^* g$.

DEFINITION OF $\mathfrak{d}$. The **dominating number** $\mathfrak{d}$ is the smallest cardinality of any dominating family; more formally

$$\mathfrak{d} = \min\big\{|\mathscr{D}| : \mathscr{D} \subseteq {}^\omega\omega \text{ is dominating}\big\}.$$

A family $\mathscr{B} \subseteq {}^\omega\omega$ is **unbounded** if there is no single function $f \in {}^\omega\omega$ which dominates all functions of $\mathscr{B}$, *i.e.*, for every $f \in {}^\omega\omega$ there is a $g \in \mathscr{B}$ such that $g \not<^* f$. Since "$<^*$" is not a linear ordering, an unbounded family is not necessarily dominating — but vice versa (see FACT 8.2).

DEFINITION OF $\mathfrak{b}$. The **bounding number** $\mathfrak{b}$ is the smallest cardinality of any unbounded family; more formally

$$\mathfrak{b} = \min\big\{|\mathscr{B}| : \mathscr{B} \subseteq {}^\omega\omega \text{ is unbounded}\big\}.$$

$\mathfrak{c} = |\{0,1\}^{\mathbb{N}}| = |\mathcal{P}(\mathbb{N})|^{\mathbb{N}}$

$\mathfrak{c}$ $=$ $|\mathbb{N}^{\mathbb{N}}|$

Obviously, the family ${}^{\omega}\omega$ itself is dominating and therefore unbounded, which shows that $\mathfrak{d}$ and $\mathfrak{b}$ are well-defined and $\mathfrak{b}, \mathfrak{d} \leq \mathfrak{c}$. Moreover, we have the following

FACT 8.2. $\mathfrak{b} \leq \mathfrak{d}$.

Proof. It is enough to show that every dominating family is unbounded. So, let $\mathscr{D} \subseteq {}^{\omega}\omega$ be a dominating family and let $f \in {}^{\omega}\omega$ be an arbitrary function. Since $\mathscr{D}$ is dominating, there is a $g \in \mathscr{D}$ such that $f <^* g$, *i.e.*, there is an $n_0 \in \omega$ such that for all $k \geq n_0$, $f(k) < g(k)$. Hence we get $g \not<^* f$, and since f was arbitrary this implies that $\mathscr{D}$ is unbounded. ⊣

It is natural to ask whether $\mathfrak{b}$ can be smaller than $\mathfrak{d}$, or at least smaller than $\mathfrak{c}$; however, the following result shows that $\mathfrak{b}$ cannot be too small.

THEOREM 8.3. $\aleph_1 \leq \mathfrak{b}$. $\aleph_0 < \mathfrak{b}$

Proof. Let $\mathscr{E} = \{g_n \in {}^{\omega}\omega : n \in \omega\}$ be a countable family. We construct a function $f \in {}^{\omega}\omega$ which dominates all functions of $\mathscr{E}$: For each $k \in \omega$ let

f g_0 g_1 g_2 $+1$

$$f(k) = \overset{\max}{\cancel{\bigcup}}\{g_i(k) : i \in k\}\,{}^{+1}.$$

Then for every $k \in \omega$ and each $i \in k$ we have $f(k) \geq g_i(k)$ which shows that for all $n \in \omega$, $g_n <^* f$, hence, f dominates all functions of $\mathscr{E}$. ⊣

One could also define dominating and unbounded families with respect to the ordering "$<$" defined by stipulating $f < g \iff \forall k \in \omega\,(f(k) < g(k))$. Then the corresponding dominating number would be the same as $\mathfrak{d}$, as any dominating family can be made dominating in the new sense by adding all finite modifications of its members; but the corresponding bounding number would drop to ω, as the family of all constant functions is unbounded (we leave the details to the reader).

y x

The Cardinals $\mathfrak{s}$ and $\mathfrak{r}$

rozdziela nsk.

A set $x \subseteq \omega$ splits an infinite set $y \in [\omega]^{\omega}$ if both $y \cap x$ and $y \setminus x$ are infinite (*i.e.*, $|y \cap x| = |y \setminus x| = \omega$). Notice that any $x \subseteq \omega$ which splits a set $y \in [\omega]^{\omega}$ must be infinite. A **splitting family** is a family $\mathscr{S} \subseteq [\omega]^{\omega}$ such that each $y \in [\omega]^{\omega}$ is split by at least one $x \in \mathscr{S}$.

DEFINITION OF $\mathfrak{s}$. The **splitting number** $\mathfrak{s}$ is the smallest cardinality of any splitting family; more formally

$$\mathfrak{s} = \min\{|\mathscr{S}| : \mathscr{S} \subseteq [\omega]^{\omega} \text{ is splitting}\}.$$

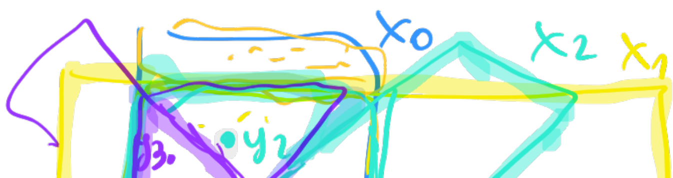
nsk. podzbiór $\mathbb{N}$

Tw. $\aleph_1 \leq \mathfrak{s}$ $(\aleph_0 < \mathfrak{s})$

T: Każda przeliczalna rodzina nsk. podzbiorów zb. $\mathbb{N}$ nie jest rozdzielająca

$S = \{X_0, X_1, \ldots\}$ $\exists\, Y \subset \mathbb{N}$, że żaden $X_0, X_1, \ldots$ nie rozdziela Y

$\{y_0, y_1, y_2, \ldots\}$



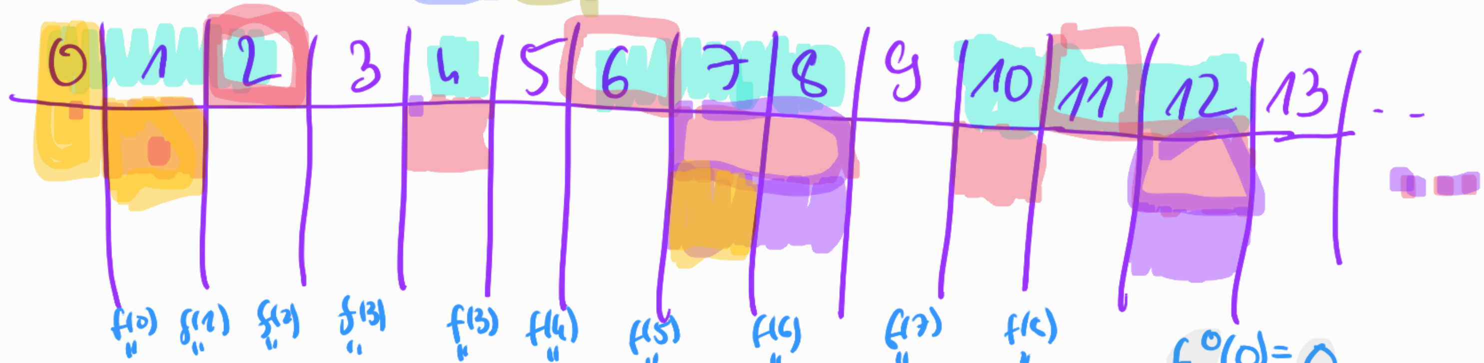
Fakt. $\lambda^1_1 \leq \mathcal{S}$.

Każda rodzina przeliczalnie wielu nieskończonych podzbiorów $\mathbb{N}$ nie jest rozdzielająca.

$\mathcal{S}$ – przeliczalna rodzina podzbiorów zb. $\mathbb{N}$

to istnieje $y \subseteq \mathbb{N}$, żaden $x \in \mathcal{S}$ nie rozdziela y

$\mathcal{S} = \{x_0, x_1, x_2, x_3, \ldots\}$



$f: 1\ 3\ 5\ 8\ 13\ 21\ 34\ 55\ 89.\ 144$

$f^0(0) = 0$

$f^1(0) = f(f(0)) = f(0) = 1$

$f^2(0) = f(f^1(0)) = f(1) = 3$

$f^3(0) = f(f^2(0)) = f(3) = 8$

$f^4(0) = f(f^3(0)) = f(8) = 144$

$f^5(0) = f(f^4(0)) = f(144)$

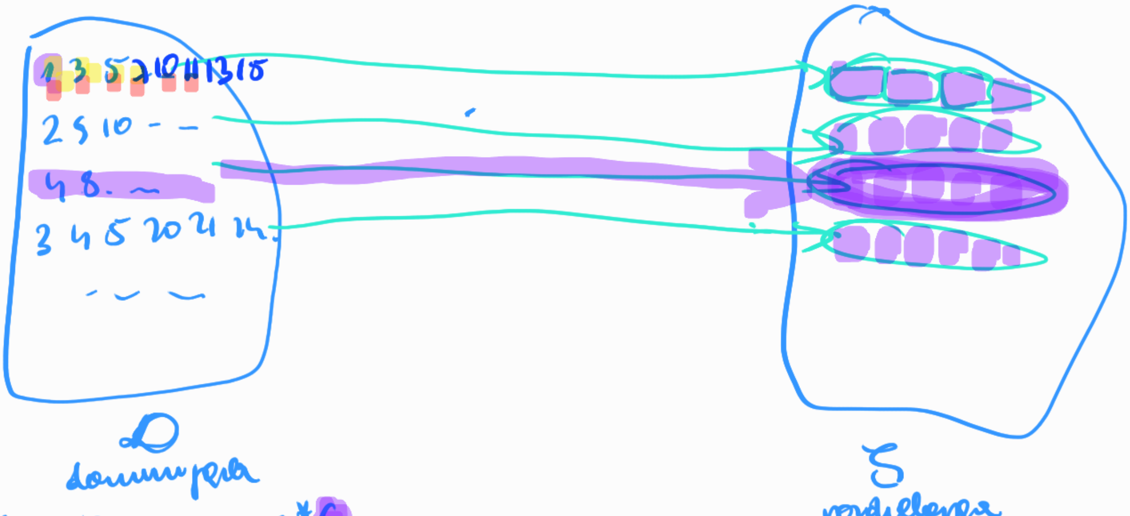
$\sigma_f: [0,1) \cup [3,8) \cup [144, \ldots)$

$\{0, 3,4,5,6,7, 144, \ldots\}$

$x: \{1,3,5,7,\ldots\}$

$f_x: 1,3,5,7,\ldots$

$k < f^k(0)$



$\forall g\ \exists f \in \mathcal{D}\ g <^* f$

By THEOREM 8.1 and later results we get $\omega_1 \leq \mathfrak{s}$ — we leave it as an exercise to the reader to find a direct proof of the uncountability of $\mathfrak{s}$.

In the proof of the following result we will see how to construct a splitting family from a dominating family.

THEOREM 8.4. $\mathfrak{s} \leq \mathfrak{d}$.

Proof. For each strictly increasing function $f \in {}^\omega\omega$ with $f(0) > 0$ let

$$\sigma_f = \bigcup \Big\{ \big[f^{2n}(0), f^{2n+1}(0)\big) : n \in \omega \Big\},$$

$f^1(0) = f(f^0(0)) = f(0)$

18:00

where for $a, b \in \omega$, $[a, b) := \{k \in \omega : a \leq k < b\}$ and $f^{n+1}(0) = f\big(f^n(0)\big)$ with $f^0(0) := 0$. Let $\mathscr{D} \subseteq {}^\omega\omega$ be a dominating family. Without loss of generality we may assume that every $f \in \mathscr{D}$ is strictly increasing and $f(0) > 0$, and let

$$\mathscr{S}_{\mathscr{D}} = \{\sigma_f : f \in \mathscr{D}\}.$$

We show that $\mathscr{S}_{\mathscr{D}}$ is a splitting family. So, fix an arbitrary $x \in [\omega]^\omega$ and let $f_x \in {}^\omega\omega$ be the (unique) strictly increasing bijection between ω and x. More formally, define $f_x : \omega \to x$ by stipulating

$$f_x(k) = \overset{\min}{\cancel{\bigcap}} \Big(x \setminus \{f_x(i) : i < k\} \Big).$$

Notice that for all $k \in \omega$, $f_x(k) \geq k$. Since $\mathscr{D}$ is dominating there is an $f \in \mathscr{D}$ such that $f_x <^* f$, which implies that there is an $n_0 \in \omega$ such that for all $k \geq n_0$ we have $f_x(k) < f(k)$. For each $k \in \omega$ we have $k \leq f^k(0)$ as well as $k \leq f_x(k)$. Moreover, for $k \geq n_0$ we have

$$f^k(0) \leq f_x\big(f^k(0)\big) < f\big(f^k(0)\big) = f^{k+1}(0)$$

and therefore $f_x\big(f^k(0)\big) \in \big[f^k(0), f^{k+1}(0)\big)$. Thus, for all $k \geq n_0$ we have $f_x\big(f^k(0)\big) \in \sigma_f$ *iff* k is even, which shows that both $x \cap \sigma_f$ and $x \setminus \sigma_f$ are infinite. Hence, σ_f splits x, and since x was arbitrary, $\mathscr{S}_{\mathscr{D}}$ is a splitting family. ⊣

kosiarka nierozdzielalne

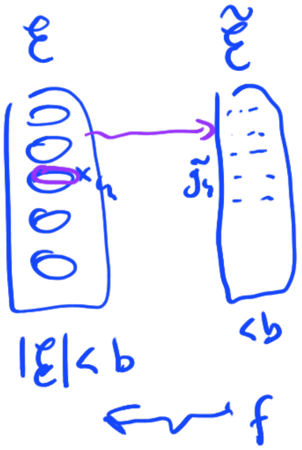
A **reaping family** — also known as *refining* or *unsplittable* family — is a family $\mathscr{R} \subseteq [\omega]^\omega$ such that there is no single set $x \in [\omega]^\omega$ which splits all elements of $\mathscr{R}$, *i.e.*, for every $x \in [\omega]^\omega$ there is a $y \in \mathscr{R}$ such that $y \cap x$ or $y \setminus x$ is finite. In other words, a family $\mathscr{R}$ is reaping if for every $x \in [\omega]^\omega$ there is a $y \in \mathscr{R}$ such that $y \subseteq^* (\omega \setminus x)$ or $y \subseteq^* x$. The origin of "reaping" in this context is that A *reaps* B *iff* A *splits* B, by analogy with a scythe cutting the stalks of grain when one reaps the grain. So, a *reaping family* would be a *splitting family*. However, the more logical approach, where "reaps" means "is unsplit by", seems to have no connection with the everyday meaning of the word "reap".

DEFINITION OF $\mathfrak{r}$. The **reaping number** $\mathfrak{r}$ is the smallest cardinality of any reaping family; more formally

$|[\omega]^\omega| = \mathfrak{c}$

$$\mathfrak{r} = \min\left\{|\mathscr{R}| : \mathscr{R} \subseteq [\omega]^\omega \text{ is reaping}\right\}.$$

Since the family $[\omega]^\omega$ is obviously reaping, $\mathfrak{r}$ is well-defined and $\mathfrak{r} \leq \mathfrak{c}$. Furthermore, by THEOREM 8.3, the following result implies that every reaping family is uncountable:



$x_5 = \{0,1,3,5,7,10,11,13,15,\dots\}$

$g_5 = 1,3,5,7,10,11,13,15$ $\longrightarrow$ $\tilde{g}_5 = 0,1,3,7,15,\dots$

THEOREM 8.5. $\mathfrak{b} \leq \mathfrak{r}$.

Proof. Let $\mathscr{E} = \{x_\xi \in [\omega]^\omega : \xi \in \kappa < \mathfrak{b}\}$ be an arbitrary family of infinite subsets of ω of cardinality strictly less than $\mathfrak{b}$. We show that $\mathscr{E}$ is not a reaping family. For each $x_\xi \in \mathscr{E}$ let $g_\xi \in {}^\omega\omega$ be the unique strictly increasing bijection between ω and $x_\xi \setminus \{0\}$. Further, let $\tilde{g}_\xi(k) := g_\xi^k(0)$, where $g_\xi^{k+1}(0) = g_\xi\big(g_\xi^k(0)\big)$ and $g_\xi^0(0) := 0$. Consider $\tilde{\mathscr{E}} = \{\tilde{g}_\xi : \xi \in \kappa\}$. Since $\kappa < \mathfrak{b}$, the family $\tilde{\mathscr{E}}$ is bounded, *i.e.*, there exists an $f \in {}^\omega\omega$ such that for all $\xi \in \kappa$, $\tilde{g}_\xi <^* f$. Let $x = \bigcup_{k\in\omega}\big[f^{2k}(0), f^{2k+1}(0)\big)$. Then for each $\xi \in \kappa$ there is an $n_\xi \in \omega$ such that for all $k \geq n_\xi$, $f^k(0) \leq g_\xi\big(f^k(0)\big) < f\big(f^k(0)\big)$. This implies that neither $x_\xi \subseteq^* x$ nor $x_\xi \subseteq^* (\omega \setminus x)$, and hence, $\mathscr{E}$ is not a reaping family. ⊣

$f^{(k+1)}(0)$

0 1 2 3 4

18:58

The Cardinals $\mathfrak{a}$ and $\mathfrak{i}$

Two sets $x, y \in [\omega]^\omega$ are **almost disjoint** if $x \cap y$ is finite. A family $\mathscr{A} \subseteq [\omega]^\omega$ of pairwise almost disjoint sets is called an **almost disjoint** family; and a **maximal almost disjoint** (*mad*) family is an infinite almost disjoint family $\mathscr{A} \subseteq [\omega]^\omega$ which is maximal with respect to inclusion, *i.e.*, $\mathscr{A}$ is not properly contained in any almost disjoint family $\mathscr{A}' \subseteq [\omega]^\omega$.

DEFINITION OF $\mathfrak{a}$. The **almost disjoint number** $\mathfrak{a}$ is the smallest cardinality of any maximal almost disjoint family; more formally

$$\mathfrak{a} = \min\left\{|\mathscr{A}| : \mathscr{A} \subseteq [\omega]^\omega \text{ is } \textit{mad}\right\}.$$

Before we show that $\mathfrak{b} \leq \mathfrak{a}$ (which implies that $\mathfrak{a}$ is uncountable), let us show first that there is a *mad* family of cardinality $\mathfrak{c}$.

PROPOSITION 8.6. *There exists a maximal almost disjoint family of cardinality* $\mathfrak{c}$.

Proof. Notice that by Teichmüller's Principle, every almost disjoint family can be extended to a *mad* family. So, it is enough to construct an almost disjoint family $\mathscr{A}_0$ of cardinality $\mathfrak{c}$. Let $\{s_i : i \in \omega\}$ be an enumeration of $\bigcup_{n\in\omega} {}^n\omega$, *i.e.*, for each $t : n \to \omega$ there is a unique $i \in \omega$ such that $t = s_i$. For $f \in {}^\omega\omega$ let

$$x_f = \left\{i \in \omega : \exists n \in \omega\,(f|_n = s_i)\right\}.$$

Then, for any distinct functions $f, g \in {}^\omega\omega$, $x_f \cap x_g$ is finite. Indeed, if $f \neq g$, then there is an $n_0 \in \omega$ such that $f(n_0) \neq g(n_0)$ which implies that for all $k > n_0$, $f|_k \neq g|_k$, and hence, $|x_f \cap x_g| \leq n_0+1$. Now, let $\mathscr{A}_0 := \{x_f : f \in {}^\omega\omega\}$. Then $\mathscr{A}_0 \subseteq [\omega]^\omega$ is a set of pairwise almost disjoint infinite subsets of ω, therefore, $\mathscr{A}_0$ is an almost disjoint family of cardinality $|{}^\omega\omega| = \mathfrak{c}$. $\dashv$

The following result implies that $\mathfrak{a}$ is uncountable and in the proof we will show how one can construct an unbounded family from a *mad* family.

THEOREM 8.7. $\mathfrak{b} \leq \mathfrak{a}$.

Proof. Let $\mathscr{A} = \{x_\xi : \xi \in \kappa\}$ be a *mad* family. It is enough to construct an unbounded family of cardinality $|\mathscr{A}|$. Let $z = \omega \setminus \bigcup_{\xi\in\kappa} x_\xi$; then z is finite (otherwise, $\mathscr{A}\cup\{z\}$ would be an almost disjoint family which properly contains $\mathscr{A}$). Let $x'_0 := x_0 \cup z \cup \{0\}$ and for positive integers $n \in \omega$ let $x'_n := \big(x_n \cup \{n\}\big) \setminus \bigcup_{k\in n} x'_k$. Then, since $\mathscr{A}$ is an almost disjoint family, $\{x'_n : n \in \omega\}$ is a family of pairwise disjoint infinite subsets of ω and by construction, $\bigcup_{n\in\omega} x'_n = \omega$. Moreover, $\big(\mathscr{A} \setminus \{x_\xi : \xi \in \omega\}\big) \cup \{x'_n : n \in \omega\}$ is still *mad*. For $n \in \omega$ let $g_n \in {}^\omega\omega$ be the unique strictly increasing bijection from x'_n to ω, and let $h : \omega \to \omega \times \omega$ defined by stipulating

$$h(m) = \langle n, k\rangle \quad \text{where } m \in x'_n \text{ and } k = g_n(m)\,.$$

By definition, for each $n \in \omega$, $h[x'_n] = \big\{\langle n, k\rangle : k \in \omega\big\}$, and for all $\xi \in \kappa$, $h[x_{\omega+\xi}] \cap x'_n$ is finite. Further, for each $\xi \in \kappa$ define $f_\xi \in {}^\omega\omega$ by stipulating

$$f_\xi(k) = \bigcup \big(h[x_{\omega+\xi}] \cap x'_k\big)$$

and let $\mathscr{B} = \{f_\xi \in {}^\omega\omega : \xi \in \kappa\}$. Then by definition $|\mathscr{B}| = |\mathscr{A}|$; moreover, $\mathscr{B}$ is unbounded. Indeed, if there would be a function $f \in {}^\omega\omega$ which dominates all functions of $\mathscr{B}$, then the infinite set $\big\{h^{-1}\big(\langle n, f(n)\rangle\big) : n \in \omega\big\}$ would have finite intersection which each element of $\mathscr{A}$ contrary to maximality of $\mathscr{A}$. $\dashv$

A family $\mathscr{I} \subseteq [\omega]^\omega$ is called **independent** if the intersection of any finitely many members of $\mathscr{I}$ and the complements of any finitely many other members of $\mathscr{I}$ is infinite. More formally, $\mathscr{I} \subseteq [\omega]^\omega$ is independent if for any $n, m \in \omega$ and disjoint sets $\{x_i : i \in n\}, \{y_j : j \in m\} \subseteq \mathscr{I}$,

$$\bigcap_{i\in n} x_i \cap \bigcap_{j\in m} (\omega \setminus y_j) \quad \text{is infinite}\,,$$

where we stipulate $\bigcap\emptyset := \omega$. Equivalently, $\mathscr{I} \subseteq [\omega]^\omega$ is independent if for any $I, J \in \operatorname{fin}(\mathscr{I})$ with $I \cap J = \emptyset$ we have

$$\bigcap I \setminus \bigcup J \quad \text{is infinite}\,.$$

We leave it as an exercise to the reader to show that if $\mathscr{I}$ is infinite, then $\mathscr{I}$ is independent *iff* for any disjoint sets $I, J \in \operatorname{fin}(\mathscr{I})$, $\bigcap I \setminus \bigcup J \neq \emptyset$.

A **maximal independent** family is an independent family $\mathscr{I} \subseteq [\omega]^\omega$ which is maximal with respect to inclusion, *i.e.*, $\mathscr{I}$ is not properly contained in any independent family $\mathscr{I}' \subseteq [\omega]^\omega$.

DEFINITION OF $\mathfrak{i}$. The **independence number $\mathfrak{i}$** is the smallest cardinality of any maximal independent family; more formally

$$\mathfrak{i} = \min\big\{|\mathscr{I}| : \mathscr{I} \subseteq [\omega]^\omega \text{ is independent}\big\}.$$

We shall see that $\max\{\mathfrak{r}, \mathfrak{d}\} \leq \mathfrak{i}$ (which implies that $\mathfrak{i}$ is uncountable), but first let us show that there is a maximal independent family of cardinality $\mathfrak{c}$.

PROPOSITION 8.8. *There is a maximal independent family of cardinality* $\mathfrak{c}$.

Proof. It is enough to construct an independent family of cardinality $\mathfrak{c}$ on some countably infinite set. So, let us construct an independent family of cardinality $\mathfrak{c}$ on the countably infinite set

$$C = \big\{\langle s, A\rangle : s \in \operatorname{fin}(\omega) \wedge A \subseteq \mathscr{P}(s)\big\}.$$

Further, for each $x \subseteq [\omega]^\omega$ define

$$P_x := \big\{\langle s, A\rangle \in C : x \cap s \in A\big\}.$$

Notice that for any distinct $x, y \in [\omega]^\omega$ there is a finite set $s \in \operatorname{fin}(\omega)$ such that $x \cap s \neq y \cap s$, and consequently we get $P_x \neq P_y$ which implies that the set $\mathscr{I}_0 = \big\{P_x : x \in [\omega]^\omega\big\} \subseteq [C]^\omega$ is of cardinality $\mathfrak{c}$. Moreover, $\mathscr{I}_0$ is an independent family on C. Indeed, for any finitely many distinct infinite subsets of ω, say $x_0, \dots, x_m, \dots, x_{m+n}$ where $m, n \in \omega$, there is a finite set $s \subseteq \omega$ such that for all i, j with $0 \leq i < j \leq m+n$ we have $x_i \cap s \neq x_j \cap s$. Let $A = \{s \cap x_i : 0 \leq i \leq m\} \subseteq \mathscr{P}(s)$, and for every $k \in \omega \setminus s$ let $s_k := s \cup \{k\}$ and $A_k := A \cup \big\{t \cup \{k\} : t \in A\big\}$. Then

$$\big\{\langle s_k, A_k\rangle : k \in \omega \setminus s\big\} \subseteq \bigcap_{0\leq i\leq m} P_{x_i} \setminus \bigcup_{1\leq j\leq n} P_{x_{m+j}},$$

which shows that $\bigcap\{P_{x_i} : 0 \leq i \leq m\} \setminus \bigcup\{P_{x_{m+j}} : 1 \leq j \leq n\}$ is infinite, and therefore, $\mathscr{I}_0$ is an independent family on C of cardinality $\mathfrak{c}$. ⊣

The following result implies that $\mathfrak{i}$ is uncountable.

THEOREM 8.9. $\max\{\mathfrak{r}, \mathfrak{d}\} \leq \mathfrak{i}$.

Proof. $\mathfrak{r} \leq \mathfrak{i}$: The idea is to show that every maximal independent family yields a reaping family of the same cardinality. For this, let $\mathscr{I} \subseteq [\omega]^\omega$ be a maximal independent family of cardinality $\mathfrak{i}$ and let

$$\mathscr{R} = \Big\{\bigcap I \setminus \bigcup J : I, J \in \operatorname{fin}(\mathscr{I}) \wedge I \cap J = \emptyset\Big\}.$$

Then $\mathscr{R}$ is a family of cardinality $\mathfrak{i}$. Furthermore, since $\mathscr{I}$ is a maximal independent family, for every $x \in [\omega]^\omega$ we find a $y \in \mathscr{R}$ (*i.e.*, $y = \bigcap I \setminus \bigcup J$) such that either $x \cap y$ or $(\omega \setminus x) \cap y$ is finite, and because $(\omega \setminus x) \cap y = y \setminus x$, this shows that x does not split all elements of $\mathscr{R}$. Thus, $\mathscr{R}$ is a reaping family of cardinality $\mathfrak{i}$, and therefore $\mathfrak{r} \leq \mathfrak{i}$.

$\mathfrak{d} \leq \mathfrak{i}$: The idea is to show that an independent family of cardinality strictly less than $\mathfrak{d}$ cannot be maximal. For this, suppose $\mathscr{I} = \{X_\xi : \xi \in \kappa < \mathfrak{d}\} \subseteq [\omega]^\omega$ is an infinite independent family of cardinality $\kappa < \mathfrak{d}$. We shall construct a set $Z \in [\omega]^\omega$ such that $\mathscr{I} \cup \{Z\}$ is still independent, which implies that the independent family $\mathscr{I}$ is not maximal. For this it is enough to show that for any finite, disjoint subfamilies of $\mathscr{I}$, say I and J, the infinite set $\bigcap I \setminus \bigcup J$ meets both Z and $\omega \setminus Z$ in an infinite set.

Let $\mathscr{I}_\omega := \{X_n : n \in \omega\} \subseteq \mathscr{I}$ be a countably infinite subfamily of $\mathscr{I}$ and for each $n \in \omega$ let $X_n^0 := X_n$ and $X_n^1 := \omega \setminus X_n$. Further, for each $g \in {}^\omega 2$ let

$$C_{n,g} = \bigcap_{k \in n} X_k^{g(k)}$$

and for $\mathscr{I}' := \mathscr{I} \setminus \mathscr{I}_\omega$ define

$$\mathscr{F} = \Big\{ \bigcap I' \setminus \bigcup J' : I' \text{ and } J' \text{ are finite, disjoint subfamilies of } \mathscr{I}' \Big\}.$$

Claim. *The family $\mathscr{C} = \{C_{n,g} : n \in \omega\}$ has a pseudo-intersection that has infinite intersection with every set in $\mathscr{F}$.*

Proof of Claim. Since $\mathscr{I}$ is an infinite independent family of cardinality $\kappa < \mathfrak{d}$, $\mathscr{F} \subseteq [\omega]^\omega$ is a family of cardinality κ such that each set in $\mathscr{F}$ has infinite intersection with every member of $\mathscr{C}$. For any $h \in {}^\omega\omega$ define

$$Y_g^h = \bigcup_{n \in \omega} \big(C_{n,g} \cap h(n)\big).$$

Since $\langle C_{n,g} : n \in \omega \rangle$ is decreasing (*i.e.*, $C_{n,g} \supseteq C_{m,g}$ whenever $n \leq m$), Y_g^h is almost contained in each member of $\mathscr{C}$ — however, Y_g^h is not necessarily infinite. It remains to choose the function $h \in {}^\omega\omega$ so that Y_g^h is infinite (*i.e.*, Y_g^h is a pseudo-intersection of $\mathscr{C}$) and has infinite intersection with every set in $\mathscr{F}$. Notice first that for every $A \in \mathscr{F}$ and for every $n \in \omega$, $A \cap C_{n,g}$ is infinite; thus, for every $A \in \mathscr{F}$ we can define a function $f_A(n) \in {}^\omega\omega$ by stipulating

$$f_A(n) = \text{the } n^{th} \text{ element (in increasing order) of } A \cap C_{n,g}.$$

Since $|\mathscr{F}| < \mathfrak{d}$, the family $\{f_A : A \in \mathscr{F}\}$ is not dominating. In particular, there is a function $h_0 \in {}^\omega\omega$ with the property that for each $A \in \mathscr{F}$ the set

$$D_A = \big\{n \in \omega : h_0(n) > f_A(n)\big\}$$

is infinite. Now, for each $A \in \mathscr{F}$ and every $n \in D_A$ we have $h_0(n) \geq f_A(n) + 1$ which implies that $|A \cap h_0(n)| \geq |A \cap f_A(n) + 1| = n$, and since D_A is infinite, also $A \cap Y_g^{h_0}$ is infinite. Finally, by construction $Y_g^{h_0}$ is a pseudo-intersection of $\mathscr{C}$ that has infinite intersection with every set in $\mathscr{F}$. $\dashv_{\text{Claim}}$

By the CLAIM, for every $g \in {}^\omega 2$ there is a set, say $Y_g \in [\omega]^\omega$, which has the following two properties:

(1) For all $n \in \omega$, $Y_g \subseteq^* \bigcap_{k\in n} X_k^{g(k)}$.

(2) $Y_g \cap \left(\bigcap I' \setminus \bigcup J'\right)$ is infinite whenever I' and J' are finite, disjoint subfamilies of $\mathscr{I}'$.

It follows from (1) that for any distinct $g, g' \in {}^\omega\omega$, Y_g and $Y_{g'}$ are almost disjoint. Let now

$$Q_0 = \left\{g \in {}^\omega\omega : \exists n_0 \in \omega\, \forall k \geq n_0 \left(g(k) = 0\right)\right\}$$

and

$$Q_1 = \left\{g \in {}^\omega\omega : \exists n_1 \in \omega\, \forall k \geq n_1 \left(g(k) = 1\right)\right\}.$$

Then $Q_0 \cup Q_1$ is a countably infinite subset of ${}^\omega\omega$. Let $\{g_n : n \in \omega\}$ be an enumeration of $Q_0 \cup Q_1$ and for each $n \in \omega$ let $Y'_{g_n} := Y_{g_n} \setminus \bigcup\{Y_{g_k} : k \in n\}$. Then $\{Y'_{g_n} : n \in \omega\}$ is a countable family of pairwise disjoint infinite subsets of ω. Finally let

$$Z = \bigcup_{g\in Q_0} Y'_g \quad \text{and} \quad Z' = \bigcup_{g\in Q_1} Y'_g .$$

Then Z and Z' are disjoint. Now we show that Z has infinite intersection with every $\bigcap I \setminus \bigcup J$, where I and J are arbitrary finite subfamilies of $\mathscr{I}$; and since the same also holds for $Z' \subseteq \omega \setminus Z$, $\mathscr{I} \cup \{Z\}$ is an independent family, *i.e.*, the independent family $\mathscr{I}$ of cardinality $< \mathfrak{d}$ is not maximal.

Given any finite, disjoint subfamilies $I, J \subseteq \mathscr{I}$, and let $I_0 = I \cap \mathscr{I}_\omega$, $J_0 = J \cap \mathscr{I}_\omega$, $I' = I \setminus I_0$, $J' = J \setminus J_0$, where $\mathscr{I}_\omega = \{X_n : n \in \omega\}$. Further, let $m \in \omega$ be such that $I_0 \cup J_0 \subseteq \{X_n : n \in m\} \subseteq \mathscr{I}_\omega$ and fix $g \in Q_0$ such that for all $n \in m$,

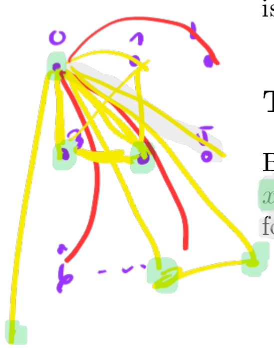
$$\left(X_n \in (I_0 \cup J_0) \wedge g(n) = 0\right) \leftrightarrow X_n \in I_0 .$$

We get the following inclusions:

$$\bigcap I \setminus \bigcup J \;\supseteq\; \left(\bigcap I' \setminus \bigcap J'\right) \cap \bigcap_{n\in m} X_n^{g(n)} \;{}^*\!\supseteq\; \left(\bigcap I' \setminus \bigcap J'\right) \cap Y_g$$

The intersection on the very right is infinite (by property (2) of Y_g) and is contained in Z (because $g \in Q_0$). Hence, we have found an infinite set which is almost contained in $Z \cap \left(\bigcap I \setminus \bigcup J\right)$, and therefore Z is infinite. $\dashv$

$\{0,5\} \mapsto 0$



The Cardinals $\mathfrak{par}$ and $\mathfrak{hom}$

$\{0,1\}$

nsk.

By RAMSEY'S THEOREM 2.1, for every colouring $\pi : [\omega]^2 \to 2$ there is an $x \in [\omega]^\omega$ which is homogeneous for π, *i.e.*, $\pi|_{[x]^2}$ is constant. This leads to the following cardinal characteristic:

homogenizing

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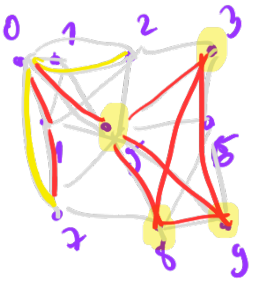
C



DEFINITION OF $\mathfrak{hom}$. The **homogeneity number** $\mathfrak{hom}$ is the smallest cardinality of any family $\mathscr{F} \subseteq [\omega]^\omega$ with the property that for every colouring $\pi : [\omega]^2 \to 2$ there is an $x \in \mathscr{F}$ which is homogeneous for π.

The following result implies that $\mathfrak{hom}$ is uncountable. In fact we will show that each family which contains a homogeneous set for every 2-colouring of $[\omega]^2$ is reaping and that each such family yields a dominating family of the same cardinality.

$\mathfrak{hom} \geq \mathfrak{r}$

$\mathfrak{hom} \geq \mathfrak{d}$

THEOREM 8.10. $\max\{\mathfrak{r}, \mathfrak{d}\} \leq \mathfrak{hom}$.

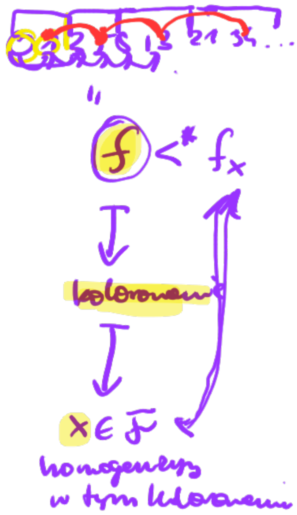
3, 5, 6, 9, ...

Proof. Let $\mathscr{F} \subseteq [\omega]^\omega$ be a family such that for every colouring $\pi : [\omega]^2 \to 2$ there is an $x \in \mathscr{F}$ which is homogeneous for π. We shall show that $\mathscr{F}$ is reaping and that $\mathscr{F}' = \{f_x \in {}^\omega\omega : x \in \mathscr{F}\}$ is dominating, where f_x is the strictly increasing bijection between ω and x.

$\mathfrak{d} \leq \mathfrak{hom}$: Firstly we show that $\mathscr{F}'$ is a dominating family. For any strictly increasing function $f \in {}^\omega\omega$ with $f(0) = 0$ define $\pi_f : [\omega]^2 \to 2$ by stipulating

$$\pi_f\big(\{n, m\}\big) = 0 \iff \exists k \in \omega\big(f(2k) \leq n, m < f(2k+2)\big).$$

Then, for every $x \in \mathscr{F}$ which is homogeneous for π_f we have $f <^* f_x$ which implies that $\mathscr{F}'$ is dominating.



$\mathfrak{r} \leq \mathfrak{hom}$: Now we show that $\mathscr{F}$ is a reaping family. Take any $y \in [\omega]^\omega$ and define $\pi_y : [\omega]^2 \to 2$ by stipulating

$y = \{0, 1, 3, 5, 8, 13, 21 \ldots\}$

$$\pi_y\big(\{n, m\}\big) = 0 \iff \{n, m\} \subseteq y \vee \{n, m\} \cap y = \emptyset.$$

Now, for every $x \in \mathscr{F}$ which is homogeneous for π_y we have either $x \subseteq y$ or $x \cap y = \emptyset$, and since y was arbitrary, $\mathscr{F}$ is reaping. ⊣

Recall that a set $H \in [\omega]^\omega$ is called **almost homogeneous** for a colouring $\pi : [\omega]^2 \to 2$ if there is a finite set $K \subseteq H$ such that $H \setminus K$ is homogeneous for π. This leads to the following cardinal characteristic:

DEFINITION OF $\mathfrak{par}$. The **partition number** $\mathfrak{par}$ is the smallest cardinality of any family $\mathscr{P}$ of 2-colourings of $[\omega]^2$ such that no single $H \in [\omega]^\omega$ is almost homogeneous for all $\pi \in \mathscr{P}$.

By PROPOSITION 2.8 we get that $\mathfrak{par}$ is uncountable, and the following result gives an upper bound for $\mathfrak{par}$.

THEOREM 8.11. $\mathfrak{par} = \min\{\mathfrak{s}, \mathfrak{b}\}$.

Proof. First we show that $\mathfrak{par} \leq \min\{\mathfrak{s}, \mathfrak{b}\}$ and then we show that $\mathfrak{par} \geq \min\{\mathfrak{s}, \mathfrak{b}\}$. $\mathfrak{par} \leq \mathfrak{s}$: Let $\mathscr{S} \subseteq [\omega]^\omega$ be a splitting family and for each $x \in \mathscr{S}$ define the colouring $\pi_x : [\omega]^2 \to 2$ by stipulating

$$\pi_x\big(\{n, m\}\big) = 0 \iff \{n, m\} \subseteq x \vee \{n, m\} \cap x = \emptyset$$

and let $\mathscr{P} = \{\pi_x : x \in \mathscr{S}\}$. Then, since $\mathscr{S}$ is splitting, no infinite set is almost homogeneous for all $\pi \in \mathscr{P}$.

$\mathfrak{par} \leq \mathfrak{b}$: Let $\mathscr{B} \subseteq {}^\omega\omega$ be an unbounded family. Without loss of generality we may assume that each $g \in \mathscr{B}$ is strictly increasing. For each $g \in \mathscr{B}$ define the colouring $\pi_g : [\omega]^2 \to 2$ by stipulating

$$\pi_g\big(\{n,m\}\big) = 0 \iff g(n) < m \quad \text{where } n < m\,.$$

Assume towards a contradiction that some infinite set $H \in [\omega]^\omega$ is almost homogeneous for all colourings in $\mathscr{P} = \{\pi_g : g \in \mathscr{B}\}$. We shall show that H yields a function which dominates the unbounded family $\mathscr{B}$, which is obviously a contradiction. Consider the function $h \in {}^\omega\omega$ which maps each natural number n to the second member of H above n; more formally, $h(n) := \min\big\{m \in H : \exists k \in H(n < k < m)\big\}$. For each $n \in \omega$ we have $n < k < h(n)$ with both k and $h(n)$ in H. By almost homogeneity of H, for each $g \in \mathscr{B}$ there is a finite set $K \subseteq \omega$ such that $H \setminus K$ is homogeneous for π_g, *i.e.*, for all $\{n,m\} \in [H \setminus K]^2$ with $n < m$ we have either $g(n) < m$ or $g(n) \geq m$. Since H is infinite, the latter case is impossible. On the other hand, the former case implies that for all $n \in H \setminus K$, $g(n) < h(n)$, hence, h dominates g and consequently h dominates each function of $\mathscr{B}$.

$\mathfrak{par} \geq \min\{\mathfrak{s}, \mathfrak{b}\}$: Suppose $\mathscr{P} = \big\{\pi_\xi : \xi \in \kappa < \min\{\mathfrak{s}, \mathfrak{b}\}\big\}$ is a family of 2-colouring of $[\omega]^2$. We shall construct a set $H \in [\omega]^\omega$ which is almost homogeneous for all colourings $\pi \in \mathscr{P}$. For each $\xi \in \kappa$ and all $n \in \omega$ define the function $f_{\xi,n} \in {}^\omega 2$ by stipulating

$$f_{\xi,n}(m) = \begin{cases} \pi_\xi\big(\{n,m\}\big) & \text{for } m \neq n, \\ 0 & \text{otherwise.} \end{cases}$$

Since $|\{f_{\xi,n} : \xi \in \kappa \wedge n \in \omega\}| = \kappa \cdot \omega = \kappa < \mathfrak{s}$, there is an infinite set $A \subseteq \omega$ on which all functions $f_{\xi,n}$ are almost constant; more formally, for each $\xi \in \kappa$ and each $n \in \omega$ there are $g_\xi(n) \in \omega$ and $j_\xi(n) \in \{0,1\}$ such that for all $m \geq g_\xi(n)$, $f_{\xi,n}(m) = j_\xi(n)$. Moreover, since $\kappa < \mathfrak{s}$ there is an infinite set $B \subseteq A$ on which each function $j_\xi \in {}^\omega 2$ is almost constant, say $j_\xi(n) = i_\xi$ for all $n \in B$ with $n \geq b_\xi$. Further, since $\kappa < \mathfrak{b}$ there is a strictly increasing function $h \in {}^\omega\omega$ which dominates each g_ξ, *i.e.*, for each $\xi \in \kappa$ there is an integer c_ξ such that for all $n \geq c_\xi$, $g_\xi(n) < h(n)$. Let $H = \{x_k : k \in \omega\} \subseteq B$ be such that for all $k \in \omega$, $h(x_k) < x_{k+1}$. Then H is almost homogeneous for each $\pi_\xi \in \mathscr{P}$. Indeed, if $n, m \in H$ are such that $\max\{b_\xi, c_\xi\} \leq n < m$, then $g_\xi(n) < h(n) < m$ and therefore $\pi_\xi\big(\{n,m\}\big) = f_{\xi,n}(m) = j_\xi(n) = i_\xi$, *i.e.*, $H \setminus \max\{b_\xi, c_\xi\}$ is homogeneous for π_ξ. ⊣

The Cardinal $\mathfrak{h}$

A family $\mathscr{H} = \{\mathscr{A}_\xi : \xi \in \kappa\} \subseteq \mathscr{P}\big([\omega]^\omega\big)$ of *mad* families of cardinality $\mathfrak{c}$ is called **shattering** if for each $x \in [\omega]^\omega$ there is a $\xi \in \kappa$ such that x has infinite intersection with at least two distinct members of $\mathscr{A}_\xi$, *i.e.*, at lest two sets of $\mathscr{A}_\xi$ split x. We leave it as an exercise to the reader to show that there are shattering families of cardinality $\mathfrak{c}$ (for each $x \in [\omega]^\omega$ take two disjoint sets $y, y' \subseteq x$ such that $\omega \setminus (y \cup y')$ is infinite and extend $\{y, y'\}$ to a *mad* family of cardinality $\mathfrak{c}$).

Definition of $\mathfrak{h}$. The **shattering number $\mathfrak{h}$** is the smallest cardinality of a shattering family; more formally

$$\mathfrak{h} = \min\left\{|\mathscr{H}| : \mathscr{H} \text{ is shattering}\right\}.$$

If one tries to visualise a shattering family, one would probably draw a kind of matrix with $\mathfrak{c}$ columns, where the rows correspond to the elements of the family (*i.e.*, to the *mad* families). Having this picture in mind, the size of the shattering family would then be the *height* of the matrix, and this where the letter "h" comes from.

In order to prove that $\mathfrak{h} \leq \mathfrak{par}$ we shall show how to construct a shattering family from any family $\mathscr{P}$ of 2-colourings of $[\omega]^2$ such that no single set is almost homogeneous for all $\pi \in \mathscr{P}$; the following lemma is the key idea in that construction:

Lemma 8.12. *For every colouring $\pi : [\omega]^2 \to 2$ there is a mad family $\mathscr{A}_\pi$ of cardinality $\mathfrak{c}$ such that each $A \in \mathscr{A}_\pi$ is homogeneous for π.*

Proof. Let $\mathscr{A} \subseteq [\omega]^\omega$ be an arbitrary almost disjoint family of cardinality $\mathfrak{c}$ and let π be a 2-colouring of $[\omega]^2$. By Ramsey's Theorem 2.1, for each $A \in \mathscr{A}$ we find an infinite set $A' \subseteq A$ such that A' is homogeneous for π. Let $\mathscr{A}' = \{A' : A \in \mathscr{A}\}$; then $\mathscr{A}'$ is an almost disjoint family of cardinality $\mathfrak{c}$ where each member of $\mathscr{A}'$ is homogeneous for π. Let $\{x_\xi : \xi \in \kappa \leq \mathfrak{c}\}$ be an enumeration of $[\omega]^\omega \setminus \mathscr{A}'$. By transfinite induction define $\mathscr{A}_0 = \mathscr{A}'$ and for each $\xi \in \kappa$ let

$$\mathscr{A}_{\xi+1} = \begin{cases} \mathscr{A}_\xi \cup \{x_\xi\} & \text{if } x_\xi \text{ is homogeneous for } \pi \text{ and} \\ & \quad \text{for each } A \in \mathscr{A}_\xi,\ x_\xi \cap A \text{ is finite,} \\ \mathscr{A}_\xi & \text{otherwise.} \end{cases}$$

By construction, $\mathscr{A}_\pi = \bigcup_{\xi\in\kappa} \mathscr{A}_\xi$ is an almost disjoint family of cardinality $\mathfrak{c}$, all whose members are homogeneous for π. Moreover, $\mathscr{A}_\pi$ is a *mad* family. Indeed, if there would be an $x \in [\omega]^\omega$ such that for all $A \in \mathscr{A}_\pi$, $x \cap A$ is finite, then, by Ramsey's Theorem 2.1, there would be an $x_{\xi_0} \in [x]^\omega$ (for some $\xi_0 \in \kappa$) which is homogeneous for π. In particular, x_{ξ_0} would belong to $\mathscr{A}_{\xi_0+1}$. Hence, $x \cap x_{\xi_0}$ is infinite, where $x_{\xi_0} \in \mathscr{A}$, which is a contradiction to the choice of x. $\dashv$

Theorem 8.13. $\mathfrak{h} \leq \mathfrak{par}$.

Proof. Let $\mathscr{P}$ be a family of 2-colourings of $[\omega]^2$ such that no single set is almost homogeneous for all $\pi \in \mathscr{P}$ and let $\mathscr{H}_{\mathscr{P}} = \{\mathscr{A}_\pi : \pi \in \mathscr{P}\}$, where $\mathscr{A}_\pi$ is like in LEMMA 8.12. We claim that $\mathscr{H}_{\mathscr{P}}$ is shattering. Indeed, let $H \subseteq \omega$ be an arbitrary infinite subset of ω. By the property of $\mathscr{P}$, there is a $\pi \in \mathscr{P}$ such that H is not almost homogeneous for π. Consider $\mathscr{A}_\pi \in \mathscr{H}_{\mathscr{P}}$: Since $\mathscr{A}_\pi$ is *mad*, there is an $A \in \mathscr{A}_\pi$ such that $H \cap A$ is infinite, and since A is homogeneous for π, $H \setminus A$ is infinite too; and again, since $\mathscr{A}_\pi$ is *mad*, there is an $A' \in \mathscr{A}_\pi$ (distinct from A) such that $(H \setminus A) \cap A'$ is infinite. This shows that H has infinite intersection with two distinct members of $\mathscr{A}_\pi$. Hence, $\mathscr{H}_{\mathscr{P}}$ is shattering. $\dashv$

In order to prove that $\mathfrak{p} \leq \mathfrak{h}$ we have to introduce some notions: If $\mathscr{A}$ and $\mathscr{A}'$ are *mad* families (of cardinality $\mathfrak{c}$), then $\mathscr{A}'$ **refines** $\mathscr{A}$, denoted $\mathscr{A}' \succ\!\!\!\succ \mathscr{A}$, if for each $A' \in \mathscr{A}'$ there is an $A \in \mathscr{A}$ such that $A' \subseteq^* A$ A shattering family $\{\mathscr{A}_\xi : \xi \in \kappa\}$ is called **refining** if $\mathscr{A}_{\xi'} \succ\!\!\!\succ \mathscr{A}_\xi$ whenever $\xi' > \xi$.

The next result is the key lemma in the proof that every shattering family of size $\mathfrak{h}$ induces a refining shattering family of the same cardinality.

LEMMA 8.14. *For every family $\mathscr{E} = \{\mathscr{A}_\xi : \xi \in \kappa < \mathfrak{h}\}$ of cardinality $\kappa < \mathfrak{h}$ of mad families of cardinality $\mathfrak{c}$ there exists a mad family $\mathscr{A}'$ which refines each $\mathscr{A}_\xi \in \mathscr{E}$. Furthermore, $\mathscr{A}'$ is of cardinality $\mathfrak{c}$.*

Proof. Let $\mathscr{E} = \{\mathscr{A}_\xi : \xi \in \kappa < \mathfrak{h}\}$ be a family of less than $\mathfrak{h}$ *mad* families of cardinality $\mathfrak{c}$. For every $x \in [\omega]^\omega$ we find an $x' \in [x]^\omega$ with the property that for each $\mathscr{A}_\xi \in \mathscr{H}$ there is an $A \in A_\xi$ such that $x' \subseteq^* A$. Indeed, if there is no such x' (for some given $x \in [\omega]^\omega$), then a bijection between x and ω would yield a shattering family of cardinality $\kappa < \mathfrak{h}$, contrary to the definition of $\mathfrak{h}$. Now, if $\mathscr{A}' \subseteq \{x' : x \in [\omega]^\omega\}$ is a *mad* family, then $\mathscr{A}'$ is of cardinality $\mathfrak{c}$ (since $\mathscr{A}_0$ is of cardinality $\mathfrak{c}$) and refines each $\mathscr{A}_\xi \in \mathscr{E}$ (since $\mathscr{A}' \subseteq \{x' : x \in [\omega]^\omega\}$). It remains to show that *mad* families $\mathscr{A}' \subseteq \{x' : x \in [\omega]^\omega\}$ exist. Indeed, if $\mathscr{A} \subseteq \{x' : x \in [\omega]^\omega\}$ is an almost disjoint family which is not maximal, then there exists an $x \in [\omega]^\omega$ such that for all $A \in \mathscr{A}$, $x \cap A$ is finite. Notice that $\mathscr{A} \cup \{x'\}$ is still an almost disjoint family, hence, by Teichmüller's Principle, every almost disjoint family $\mathscr{A} \subseteq \{x' : x \in [\omega]^\omega\}$ can be extended to a *mad* family $\mathscr{A}' \subseteq \{x' : x \in [\omega]^\omega\}$. $\dashv$

PROPOSITION 8.15. *If $\mathscr{H} = \{\mathscr{A}_\xi : \xi \in \mathfrak{h}\}$ is a shattering family of cardinality $\mathfrak{h}$, then there exists a refining shattering family $\mathscr{H}' = \{\mathscr{A}'_\xi : \xi \in \mathfrak{h}\}$ such that for each $\xi \in \mathfrak{h}$ we have $\mathscr{A}'_\xi \succ\!\!\!\succ \mathscr{A}_\xi$.*

Proof. The proof is by transfinite induction: Let $\mathscr{A}'_0 := \mathscr{A}_0$ and assume we have already defined $\mathscr{A}'_\xi$ for all $\xi \in \eta$ where $\eta \in \mathfrak{h}$. Apply LEMMA 8.14 to the family $\{\mathscr{A}'_\xi : \xi \in \eta\} \cup \{\mathscr{A}_\eta\}$ to obtain $\mathscr{A}'_\eta$ and let $\mathscr{H}' = \{\mathscr{A}'_\xi : \xi \in \mathfrak{h}\}$. $\dashv$

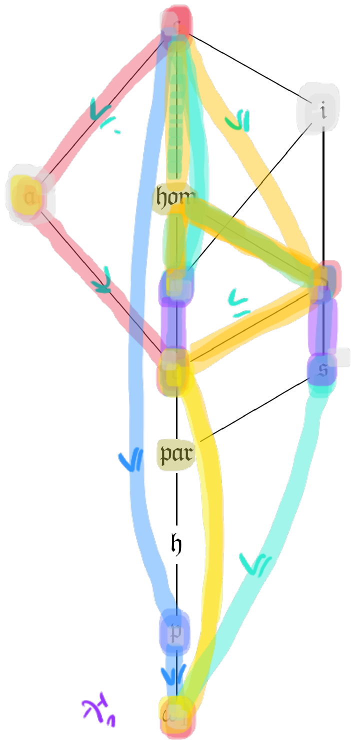
Now, the proof of $\mathfrak{p} \leq \mathfrak{h}$ is straightforward.

THEOREM 8.16. $\mathfrak{p} \leq \mathfrak{h}$.

Proof. By PROPOSITION 8.15 there exists a refining shattering family $\mathscr{H} = \{\mathscr{A}_\xi : \xi \in \mathfrak{h}\}$ of cardinality $\mathfrak{h}$. With $\mathscr{H}$ we shall build a family $\mathscr{F} \subseteq [\omega]^\omega$ of cardinality $\mathfrak{h}$ which has the *sfip* but which does not have a pseudo-intersection: Chose any $x_0 \in \mathscr{A}_0$ and assume we have already chosen $x_\xi \in \mathscr{A}_\xi$ for all $\xi \in \eta$ where $\eta \in \mathfrak{h}$. Since $\mathscr{H}$ is refining we can chose a $x_\eta \in \mathscr{A}_\eta$ such that x_η is a pseudo-intersection of $\{x_\xi : \xi \in \eta\}$. Finally let $\mathscr{F} = \{x_\xi : \xi \in \mathfrak{h}\}$. Then $\mathscr{F}$ is a family of cardinality $\leq \mathfrak{h}$ which has the *sfip*, but since $\mathscr{H}$ is shattering, no infinite set is almost contained in every member of $\mathscr{F}$, *i.e.*, $\mathscr{F}$ does not have a pseudo-intersection. ⊣

Summary

The diagram below shows the relations between the twelve cardinals. A line connecting two cardinals indicates that the cardinal lower on the diagram is less than or equal to the cardinal higher on the diagram (provably in ZFC).



Later we shall see that each of following relations is consistent with ZFC:

- $\mathfrak{a} < \mathfrak{c}$ (PROPOSITION 18.5)
- $\mathfrak{i} < \mathfrak{c}$ (PROPOSITION 18.11)
- $\omega_1 < \mathfrak{p} = \mathfrak{c}$ (PROPOSITION 19.1)

- $\mathfrak{a} < \mathfrak{d} = \mathfrak{r}$ (COROLLARY 21.11)
- $\mathfrak{s} = \mathfrak{b} < \mathfrak{d}$ (PROPOSITION 21.13)
- $\mathfrak{d} < \mathfrak{r}$ (PROPOSITION 22.4)
- $\mathfrak{d} > \mathfrak{r}$ (PROPOSITION 23.7)
- $\mathfrak{p} < \mathfrak{h}$ (PROPOSITION 24.12)

NOTES

Most of the classical cardinal characteristics and their relations presented here can be found for example in van Douwen [42] and Vaughan [43], where one finds also a few historical notes (for $\mathfrak{d}$ see also Kanamori [27, p. 179 f.]). PROPOSITION 8.8 is due to Fichtenholz and Kantorovitch [22], but the proof we gave is Hausdorff's, who generalised in [26] the result to arbitrary infinite cardinals (see also Exercise (A6) on p. 288 of Kunen [29]). THEOREM 8.9 is due to Shelah [33], however, the proof is taken from Blass [5] (see also [4, Theorem 21]), where the claim in the proof is due to Ketonen [28, Proposition 1.3]. THEOREM 8.10 and THEOREM 8.11 are due to Blass and the proofs are taken from Blass [5] (see also [4, Section 6]). The shattering cardinal $\mathfrak{h}$ was introduced and investigated by Balcar, Pelant, and Simon in [2] (*cf.* RELATED RESULT 51).

RELATED RESULTS

50. *The Continuum Hypothesis.* There are numerous statements from areas like Algebra, Combinatorics, or Topology, which are equivalent to CH. For example Erdős and Kakutani showed that CH is equivalent to the statement that $\mathbb{R}$ is the union of countably many sets of rationally independent numbers (*cf.* [20, Theorem 2]). Many more equivalents to CH can be found in Sierpiński [39]. For the historical background of CH we refer the reader to Felgner [21].

51. *On the shattering number* $\mathfrak{h}$. Balcar, Pelant, and Simon showed that $\mathfrak{h} \leq \operatorname{cf}(\mathfrak{c})$ (see [2, Theorem 4.2]), gave a direct prove for $\mathfrak{h} \leq \mathfrak{b}$ (see [2, Theorem 4.5]) and for $\mathfrak{h} \leq \mathfrak{s}$ (follows from [2, Lemma 2.11.(c)]), and showed that $\mathfrak{h}$ is regular (see [2, Lemma 2.11.(b)]. Furthermore, Lemma 2.11.(c) of Balcar, Pelant, and Simon [2] states that there are shattering families of size $\mathfrak{h}$ which have a very strong combinatorial property:

    BASE MATRIX LEMMA. *There exists a shattering family $\mathscr{H} = \{\mathscr{A}_\xi \subseteq [\omega]^\omega : \xi \in \mathfrak{h}\}$ which has the property that for each $X \in [\omega]^\omega$ there is a $\xi \in \mathfrak{h}$ and an $A \in \mathscr{A}_\xi$ such that $A \subseteq^* X$.*

    Proof. Let $\mathscr{F} = \{\mathscr{A}_\xi \subseteq [\omega]^\omega : \xi \in \mathfrak{h}\}$ be an arbitrary but fixed refining shattering family of cardinality $\mathfrak{h}$. We first prove the following

    CLAIM. *For every infinite set $X \in [\omega]^\omega$ there exists an ordinal $\bar{\xi} \in \mathfrak{h}$ such that $\left|\{C \in \mathscr{A}_{\bar{\xi}} : |C \cap X| = \omega\}\right| = \mathfrak{c}$.*

    Proof of Claim. Let $X \in [\omega]^\omega$ be an arbitrary infinite subset of ω. Firstly we show that there exists a strictly increasing sequence $\langle \xi_n : n \in \omega \rangle$ in $\mathfrak{h}$, such that for each $n \in \omega$ and $f \in {}^n2$ we find a set $C_f \in \mathscr{A}_{\xi_n}$ with the following properties:

- $|C_f \cap X| = \omega$,
- if $f, f' \in {}^n 2$ are distinct, then $C_f \neq C_{f'}$, and
- for all $f \in {}^n 2$ and $m \in n$, $C_f \subseteq^* C_{f|_m}$.

The sequence $\langle \xi_n : n \in \omega \rangle$ is constructed by induction on n: First we choose an arbitrary $\xi_0 \in \mathfrak{h}$. Now, suppose we have already found $\xi_n \in \mathfrak{h}$ for some $n \in \omega$. Since $\mathscr{F}$ is a shattering family, for every $h \in {}^n 2$ there exists a $\zeta_h > \xi_n$ such that the infinite set $C_h \cap X$ has infinite intersection with at least two members of $\mathscr{A}_{\zeta_h}$. Let $\xi_{n+1} = \bigcup \{\zeta_h : h \in {}^n 2\}$. Then, since $\mathscr{F}$ is refining, we find a family $\{C_f : f \in {}^{n+1} 2\} \subseteq \mathscr{A}_{\xi_{n+1}}$ with the desired properties.

Let $\bar{\xi} := \bigcup_{n \in \omega} \xi_n$; then the ordinal $\bar{\xi}$ is smaller than $\mathfrak{h}$: Otherwise, since $\mathscr{F}$ is refining, the family $\{\mathscr{A}_{\xi_n} : n \in \omega\}$ would be a shattering family of cardinality ω, contradicting the fact that $\mathfrak{h} \geq \omega_1$.

By construction, for each $f \in {}^\omega 2$ we find a $C_f \in \mathscr{A}_{\bar{\xi}}$ such that $C_f \cap X$ is infinite (notice that for each $n \in \omega$, $|C_{f|_n} \cap X| = \omega$), and since $\mathscr{F}$ is refining we have $C_f \neq C_{f'}$ whenever $f, f' \in {}^\omega 2$ are distinct. Thus, $\big|\{C_f \in \mathscr{A}_{\bar{\xi}} : f \in {}^\omega 2\}\big| = \mathfrak{c}$ and for each $f \in {}^\omega 2$ we have $|C_f \cap X| = \omega$. $\dashv_{\text{Claim}}$

Now we construct the shattering family $\mathscr{H} = \{\mathscr{A}_\xi \subseteq [\omega]^\omega : \xi \in \mathfrak{h}\}$ as follows: For each $\xi \in \mathfrak{h}$, let $\mathscr{X}_\xi$ be the family of all $X \in [\omega]^\omega$ such that

$$\big|\{C \in \mathscr{A}_\xi : |C \cap X| = \omega\}\big| = \mathfrak{c}\,.$$

If $\mathscr{X}_\xi = \emptyset$, then let $\mathscr{A}_\xi = \mathscr{A}_\xi$. Otherwise, define (*e.g.*, by transfinite induction) an injection $g_\xi : \mathscr{X}_\xi \hookrightarrow \mathscr{A}_\xi$ such that for each $X \in \mathscr{X}_\xi$, $\big|X \cap g_\xi(X)\big| = \omega$. Now, for each $C \in \mathscr{A}_\xi$, let $\mathscr{C}_C \subseteq [C]^\omega$ be an almost disjoint family such that $\bigcup \mathscr{C}_C = C$, and whenever $C = g_\xi(X)$ for some $X \in \mathscr{X}_\xi$ (*i.e.*, $|X \cap C| = \omega$), then there exists an $A \in \mathscr{C}_C$ with $A \subseteq^* X$. Let $\mathscr{A}_\xi := \{A \in \mathscr{C}_C : C \in \mathscr{A}_\xi\}$ and let $\mathscr{H} := \{\mathscr{A}_\xi : \xi \in \mathfrak{h}\}$. Then, by construction, for every $X \in [\omega]^\omega$ we find an ordinal $\xi \in \mathfrak{h}$ and an infinite set $A \in \mathscr{A}_\xi$ such that $A \subseteq^* X$. $\dashv$

52. *The tower number* $\mathfrak{t}$.* A family $\mathscr{T} = \{T_\alpha : \alpha \in \kappa\} \subseteq [\omega]^\omega$ is called a **tower** if $\mathscr{T}$ is well-ordered by ${}^*\!\supseteq$ (*i.e.*, $T_\beta \subseteq^* T_\alpha \leftrightarrow \alpha < \beta$) and does not have a pseudo-intersection. The **tower number** $\mathfrak{t}$ is the smallest cardinality (or height) of a tower. Obviously we have $\mathfrak{p} \leq \mathfrak{t}$ and the proof of THEOREM 8.16 shows that $\mathfrak{t} \leq \mathfrak{h}$. However, it is open whether $\mathfrak{p} < \mathfrak{t}$ is consistent with ZFC (for partial results see for example van Douwen [42], Blass [5], or Shelah [35]).

53. *A linearly ordered subset of* $[\omega]^\omega$ *of size* $\mathfrak{c}$. Let $\{q_n \in \mathbb{Q} : n \in \omega\}$ be an enumeration of the rational numbers $\mathbb{Q}$ and for every real number $r \in \mathbb{R}$ let $C_r := \{n \in \omega : q_n \leq r\}$. Then, for any real numbers $r_0 < r_1$ we have $C_{r_0} \subsetneq C_{r_1}$ and $|C_{r_1} \setminus C_{r_0}| = \omega$. Thus, with respect to the ordering "$\subsetneq$", $\{C_r : r \in \mathbb{R}\} \subseteq [\omega]^\omega$ is a linearly ordered set of size $\mathfrak{c}$. In general one can show that whenever M is infinite, the partially ordered set $\big(\mathscr{P}(M), \subsetneq\big)$ contains a linearly ordered subset of size strictly greater than $|M|$.

54. *The σ-reaping number* $\mathfrak{r}_\sigma$.* A family $\mathscr{R} \subseteq [\omega]^\omega$ is called **σ-reaping** if no countably many sets suffice to split all members of $\mathscr{R}$. The **σ-reaping number** $\mathfrak{r}_\sigma$ is the smallest cardinality of any σ-reaping family (for a definition of $\mathfrak{r}_\sigma$ in terms of bounded sequences see Vojtáš [44]). Obviously we have $\mathfrak{r} \leq \mathfrak{r}_\sigma$, but it is not known whether $\mathfrak{r} = \mathfrak{r}_\sigma$ is provable in ZFC, *i.e.*, it is not known whether $\mathfrak{r} < \mathfrak{r}_\sigma$ is consistent with ZFC (see also Vojtáš [44] and Brendle [8]).

55. *On* $\mathfrak{i}$ *and* $\mathfrak{hom}$*. We have seen that $\max\{\mathfrak{r},\mathfrak{d}\} \leq \mathfrak{hom}$ (see THEOREM 8.10) and that $\max\{\mathfrak{r},\mathfrak{d}\} \leq \mathfrak{i}$ (see THEOREM 8.9). Moreover, Blass [4, Section 6] showed that $\mathfrak{hom} = \max\{\mathfrak{r}_\sigma,\mathfrak{d}\}$ (see also Blass [5]). Thus, in every model in which $\mathfrak{r} = \mathfrak{r}_\sigma$ we have $\mathfrak{hom} \leq \mathfrak{i}$. Furthermore, one can show that $\mathfrak{hom} < \mathfrak{i}$ is consistent with ZFC: In Balcar, Hernández-Hernández, and Hrušák [1] it is shown that $\max\{\mathfrak{r}, cof(\mathscr{M})\} \leq \mathfrak{i}$, where $cof(\mathscr{M})$ is the *cofinality* of the ideal of meagre sets. On the other hand, it is possible to construct models in which $\mathfrak{d} = \mathfrak{r}_\sigma = \omega_1$ and $cof(\mathscr{M}) = \omega_2 = \mathfrak{c}$ (see for example Shelah and Zapletal [36] or Brendle and Khomskii [15]). Thus, in such models we have $\omega_1 = \mathfrak{hom} < \mathfrak{i} = \omega_2$. However, it is open whether $\mathfrak{i} < \mathfrak{hom}$ (which would imply $\mathfrak{r} < \mathfrak{r}_\sigma$) is consistent with ZFC.

56. *The ultrafilter number* $\mathfrak{u}$. A family $\mathscr{F} \subseteq [\omega]^\omega$ is a **base for an ultrafilter** $\mathscr{U} \subseteq [\omega]^\omega$ if $\mathscr{U} = \{y \in [\omega]^\omega : \exists x \in \mathscr{F}(x \subseteq y)\}$. The **ultrafilter number** $\mathfrak{u}$ is the smallest cardinality of any ultrafilter base. We leave it as an exercise to the reader to show that $\mathfrak{r} \leq \mathfrak{u}$.

57. *Consistency results.* The following statements are consistent with ZFC:
    - $\mathfrak{r} < \mathfrak{u}$ (*cf.* Goldstern and Shelah [23])
    - $\mathfrak{u} < \mathfrak{d}$ (*cf.* Blass and Shelah [6] or see Chapter 23 | RELATED RESULT 130)
    - $\mathfrak{u} < \mathfrak{a}$ (*cf.* Shelah [34], see also Brendle [13])
    - $\mathfrak{h} < \mathfrak{par}$ (*cf.* Shelah [32, Theorem 5.2] or Dow [19, Proposition 2.7])
    - $\mathfrak{hom} < \mathfrak{c}$ (see Chapter 23 | RELATED RESULT 138)
    - $\mathfrak{d} < \mathfrak{a}$ (*cf.* Shelah [34], see also Brendle [10])
    - $\omega_1 = \mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \mathfrak{d} = \omega_2$ (*cf.* Shelah [32, Sections 1 & 2])
    - $\kappa = \mathfrak{b} = \mathfrak{a} < \mathfrak{s} = \lambda$ for any regular uncountable cardinals $\kappa < \lambda$ (*cf.* Brendle and Fischer [14])
    - $\mathfrak{b} = \kappa < \kappa^+ = \mathfrak{a} = \mathfrak{c}$ for $\kappa > \omega_1$ (*cf.* Brendle [7])
    - $\omega_1 = \mathfrak{s} < \mathfrak{b} = \mathfrak{d} = \mathfrak{r} = \mathfrak{a} = \omega_2$ (*cf.* Shelah [32, Section 4])
    - $\mathrm{cf}(\mathfrak{a}) = \omega$ (*cf.* Brendle [11])
    - $\mathfrak{h} = \omega_2 +$ *there are no towers of height* ω_2 (*cf.* Dordal [17]).

    Some more results can be found for example in Blass [5], Brendle [9, 12], van Douwen [42], Dow [19], and Dordal [18].

58. *Combinatorial properties of maximal almost disjoint families.* An uncountable set of reals is a σ-set if every relative Borel subset is a relative G_δ set. Brendle and Piper showed in [16] that CH implies the existence of a *mad* family which is also a σ-set (in that paper, they also discuss related results assuming Martin's Axiom).

59. *Applications to Banach space theory.* Let $\ell_p(\kappa)$ denote the Banach space of bounded functions $f : \kappa \to \mathbb{R}$ with finite ℓ_p-norm, where for $1 \leq p < \infty$,

$$\|f\| = \sqrt[p]{\sum_{\alpha\in\kappa} |f(\alpha)|^p}\,,$$

and for $p = \infty$,

$$\|f\| = \sup\{|f(\alpha)| : \alpha \in \kappa\}\,.$$

As mentioned above, Hausdorff generalised PROPOSITION 8.8 to arbitrary infinite cardinals κ, *i.e.*, if κ is an infinite cardinal then there are independent families on κ of cardinality 2^κ. Now, using independent families on κ of cardinality 2^κ it is quite straightforward to show that $\ell_\infty(\kappa)$ contains an isomorphic

copy of $\ell_1(2^\kappa)$ (the details are left to the reader), and Halbeisen [24] showed that the dual of $\ell_\infty(\kappa)$ contains an isomorphic copy of $\ell_2(2^\kappa)$ (for an analytic proof in the case $\kappa = \omega$ see Rosenthal [31, Proposition 3.4]).
We have seen that there are almost disjoint families on ω of cardinality $\mathfrak{c} = 2^{\aleph_0}$. Unlike for independent families, this result cannot be generalised to arbitrary cardinals κ, *i.e.*, it is consistent with ZFC that for some infinite κ, there no almost disjoint family on κ of cardinality 2^κ (*see* Baumgartner[3, Theorem 5.6 (b)]). However, one can prove that for all infinite cardinals κ there is an almost disjoint family on κ of cardinality $> \kappa$ (*cf.* Tarski [41], Sierpiński [37, 38] or [40, p. 448 f.], or Baumgartner [3, Theorem 2.8]). Using an almost disjoint family of cardinality $> \kappa$ it is not hard to show that every infinite dimensional Banach space of cardinality κ has more than κ pairwise almost disjoint normalised Hamel bases (*cf.* Halbeisen [25]), and Pełczyński and Sudakov [30] showed that $c_0(\kappa)$, which is a subspace of $\ell_\infty(\kappa)$, is not complemented in $\ell_\infty(\kappa)$.

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