Seminarium z teorii zbiorów i kombinatoryki nieskończonej, 2020/2021
spotkanie 9. Aksjomaty teorii mnogości – rozwiązania

17 grudnia 2020

1. Verify that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.
2. Prove, that there is no set X such that $P(X) \subseteq X$
3. Prove that if X is inductive, then the set $Y = \{x \in X : x \subseteq X\}$ is inductive. Hence $\mathbb{N}$ (the smallest inductive set) is transitive and for each $n \in \mathbb{N}$, $n = \{m \in \mathbb{N} : m < n\}$.
4. Prove that if X is inductive, then the set Y of those elements $x \in X$ that are transitive is inductive. Hence every $n \in \mathbb{N}$ is transitive.
5. Prove, that if X is inductive, then Y the set of those transitive $x \in X$, that $x \notin x$ is inductive. Hence for all $n \in \mathbb{N}$ $n \notin n$ and $n \neq n + 1$.
6. Prove that if X is inductive then the set Y of those $x \in X$ that are transitive and every nonempty $z \subseteq x$ has $\in$-minimal element is inductive.
7. Prove that for every nonempty $X \subseteq \mathbb{N}$ X has a $\in$-minimal element.
8. Prove that if X is inductive, then so is $Y = \{x \in X : x = \varnothing \vee \exists_{z \in X} x = z \cup \{z\}\}$. Hence each $n \neq 0$ is $m + 1$ for some n.
9. Let $A \subseteq \mathbb{N}$ such that $\varnothing \in A$ and if $n \in A$, then $n + 1 \in A$. Prove that $A = \mathbb{N}$.
10. Prove that every $n \in \mathbb{N}$ is T-finite.
11. Prove that $\mathbb{N}$ is T-infinite.
12. Prove that every finite set is T-finite.
13. Prove that every infinite set is T-infinite.
14. Prove that the Separation Axiom follows from the Replacement Schema.
15. Instead of Union, Power Set and Replacement Axioms consider the following weaker versions:
    - $\forall X \exists Y (\forall x \in X)(\forall u \in x) u \in Y$,
    - $\forall X \exists Y \forall u (u \subseteq X \rightarrow u \in Y)$,
    - $\forall x \forall y \forall z (\varphi(x, y, p) \wedge \varphi(x, z, p) \rightarrow y = z) \rightarrow \forall X \exists Y ((\exists x \in X) \varphi(x, y, p) \rightarrow y \in Y)$

    Prove Union, Power Set and Replacement Axioms from these versions using Separation Schema.