

Matematyka dla Wydziału Geologii, semestr 2., 2019/2020
ćwiczenia 11. – rozwiązania

6 i 7 maja 2020

1. Oblicz:

a) $\int e^{\sqrt{x}} dx$,

Podstawiamy $t = \sqrt{x}$, zatem $x = t^2$ oraz $dx = 2t dt$. Potem przez części

$$\int e^{\sqrt{x}} dx = 2 \int t e^t dt = 2(t e^t - \int e^t dt) = 2e^t(t - 1) + C = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C.$$

b) $\int \frac{\sqrt[6]{x} dx}{1 + \sqrt[3]{x}}$.

Podstawiamy $t = \sqrt[6]{x}$, więc

$$\begin{aligned} \int \frac{\sqrt[6]{x} dx}{1 + \sqrt[3]{x}} &= \int \frac{6t^7}{1 + t^2} dt = \int (t^5 - t^3 + t) dt + \int \frac{t dt}{1 + t^2} = \\ &= t^6/6 - t^4/4 + t^2/2 + \frac{1}{2} \ln(t^2 + 1) + C = x/6 - t^{2/3}/4 + t^{1/3}/2 + \frac{1}{2} \ln(t^{1/3} + 1) + C. \end{aligned}$$

2. Oblicz:

a) $\int \frac{2 dx}{(x - 4)^4}$,

$$\int \frac{2 dx}{(x - 4)^4} \stackrel{\boxed{t = x - 4, \frac{dt}{dx} = 1}}{=} \int \frac{2 dt}{t^4} = \frac{-2}{3t^3} + C = \frac{-2}{3(x - 4)^3} + C.$$

b) $\int \frac{dx}{x^4 + x}$,

Rozkładamy mianownik na czynniki: $(x^4 + x) = x(x + 1)(x^2 - x + 1)$ i wyliczamy rozkład na ułamki proste:

$$\frac{1}{x^4 + x} = \frac{A}{x} + \frac{B}{x + 1} + \frac{Cx + D}{x^2 - x + 1},$$

a zatem:

$$\begin{aligned} 1 &= A(x + 1)(x^2 - x + 1) + Bx(x^2 - x + 1) + (Cx + D)x(x + 1) = \\ &= A(x^3 + 1) + B(x^3 - x^2 + x) + C(x^3 + x^2) + D(x^2 + x), \end{aligned}$$

czyli:

$$\begin{cases} A = 1 \\ B + D = 0 \\ -B + C + D = 0 \\ A + B + C = 0 \end{cases}.$$

Czyli: $A = 1, B = -\frac{1}{3}, C = -\frac{2}{3}, D = \frac{1}{3}$, a więc:

$$\int \frac{dx}{x^4 + x} = \int \frac{dx}{x} - \frac{1}{3} \int \frac{dx}{x + 1} - \frac{1}{3} \int \frac{(2x - 1)dx}{x^2 - x + 1} \stackrel{\boxed{t = x^2 - x + 1, \frac{dt}{dx} = 2x - 1}}{=}$$

$$= \ln|x| - \frac{1}{3} \ln|x+1| - \frac{1}{3} \ln|t| + C = \ln|x| - \frac{1}{3} \ln|x+1| - \frac{1}{3} \ln(x^2 - x + 1) + C = \ln \frac{|x|}{\sqrt[3]{|x+1|(x^2 - x + 1)}} + C.$$

c) $\int \frac{2x+3}{(x^2+2x+4)^2} dx,$

x^2+2x+4 nie ma żadnych pierwiastków, więc mamy do czynienia już z ułamkiem prostym. Więc:

$$\int \frac{2x+3}{(x^2+2x+4)^2} dx = \int \frac{2x+2}{(x^2+2x+4)^2} dx + \int \frac{1}{(x^2+2x+4)^2} dx.$$

Pierwsza z tych całek to proste podstawienie, a więc:

$$\int \frac{2x+2}{(x^2+2x+4)^2} dx \overset{\boxed{t = x^2+2x+4, \frac{dt}{dx} = 2x+2}}{=} \frac{-1}{x^2+2x+4} + C.$$

Z drugą całką jest więcej kłopotu, bo potrzebne jest wymyślne podstawienie $t = \frac{x+p/2}{\sqrt{-\Delta/4}}$, a w naszym wypadku $\Delta = 4 - 16 = -12$ oraz $p = 2$, czyli $t = \frac{x+1}{\sqrt{3}}$ i $\frac{dt}{dx} = \frac{1}{\sqrt{3}}$. Czyli:

$$\int \frac{1}{(x^2+2x+4)^2} dx = \int \frac{\sqrt{3} dt}{9(t^2+1)^2} = \frac{\sqrt{3}}{9} \int \frac{dt}{(t^2+1)^2},$$

i korzystamy z wzoru rekurencyjnego i dostajemy:

$$\frac{\sqrt{3}}{9} \int \frac{dt}{(t^2+1)^2} = \frac{\sqrt{3}}{18} \left(\frac{t}{1+t^2} + \int \frac{dt}{1+t^2} \right) = \frac{\sqrt{3}}{18} \left(\frac{t}{1+t^2} + \arctg t \right) + C,$$

a więc ostatecznie:

$$\int \frac{2x+3}{(x^2+2x+4)^2} dx = -\frac{1}{x^2+2x+4} + \frac{1}{6} \frac{x+1}{x^2+2x+4} + \frac{\sqrt{3}}{18} \arctg \left(\frac{x+1}{\sqrt{3}} \right) + C.$$

d) $\int \frac{x^2-3x+2}{x(x^2+2x+1)} dx$

Mamy:

$$\int \frac{x^2-3x+2}{x(x^2+2x+1)} dx = \int \frac{x^2+2x+1-5x+1}{x(x^2+2x+1)} dx = \int \frac{dx}{x} - \int \frac{5x-1}{x(x^2+2x+1)} dx.$$

O ile z pierwszą całką nie ma kłopotu, bo, to trzeba drugą rozłożyć na ułamki proste:

$$\frac{5x-1}{x(x^2+2x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+1}.$$

A więc:

$$\begin{cases} A = -1 \\ 2A + C = 5 \\ A + B = 0 \end{cases},$$

a zatem $A = -1, B = 1, C = 7$ i:

$$\begin{aligned} & \int \frac{5x-1}{x(x^2+2x+1)} dx = - \int \frac{dx}{x} + \int \frac{x+7}{(x+1)^2} dx = \\ & = -\ln|x| + \frac{1}{2} \int \frac{2x+2}{x^2+2x+1} dx + 6 \int \frac{dx}{(x+1)^2} = -\ln|x| + \frac{\ln(x^2+2x+1)}{2} - \frac{6}{1+x} + C. \end{aligned}$$

Zatem ostatecznie:

$$\int \frac{x^2-3x+2}{x(x^2+2x+1)} dx = \frac{\ln(x^2+2x+1)}{2} - \frac{6}{1+x} + C.$$

e) $\int \frac{3x}{x^3-1} dx,$

Mamy: $x^3 - 1 = (x-1)(x^2 + x + 1)$, a zatem:

$$\frac{3x}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1},$$

czyli:

$$\begin{cases} A - C = 0 \\ A - B + C = 3 \\ A + B = 0 \end{cases},$$

zatem: $A = 1, B = -1, C = 1$ i

$$\begin{aligned} \int \frac{3x}{x^3-1} dx &= \int \frac{dx}{x-1} + \int \frac{-x+1}{x^2+x+1} dx = \\ &= \ln|x-1| - \frac{1}{2} \int \frac{2x+1}{x^2+x+1} dx + \frac{3}{2} \int \frac{dx}{x^2+x+1} = \ln|x-1| - \frac{\ln(x^2+x+1)}{2} + \frac{3}{2} \int \frac{dx}{x^2+x+1}, \end{aligned}$$

a w przypadku tej ostatniej całki potrzebujemy naszego podstawienia: $t = \frac{x+p/2}{\sqrt{-\Delta/4}}$, a w naszym wypadku $\Delta = 1 - 4 = -3$ oraz $p = 1$, czyli $t = \frac{2x+1}{\sqrt{3}}$ i $\frac{dt}{dx} = \frac{2}{\sqrt{3}}$. Czyli:

$$\int \frac{dx}{x^2+x+1} = \int \frac{\sqrt{3}/2}{\frac{3}{4}(t^2+1)} dt = \frac{2\sqrt{3}}{3} \arctgt = \frac{2\sqrt{3}}{3} \arctg \frac{2x+1}{\sqrt{3}} + C,$$

a zatem w końcu:

$$\begin{aligned} \int \frac{3x}{x^3-1} dx &= \\ &= \ln|x-1| - \frac{\ln(x^2+x+1)}{2} + \frac{3}{2} \frac{2\sqrt{3}}{3} \arctg \frac{2x+1}{\sqrt{3}} + C. \end{aligned}$$

3. Oblicz:

a) $\int \frac{x + \sqrt{2x-3}}{x-1} dx.$

$$\begin{aligned} \int \frac{x + \sqrt{2x-3}}{x-1} dx &\overset{\boxed{t = \sqrt{2x-3}, x = \frac{t^2+3}{2}, \frac{dt}{dx} = 1/\sqrt{2x-3}}}{=} \\ &= \int \frac{t^3 + 2t^2 + 3t}{t^2+1} dt = \int \left(t + 2 + \frac{2t}{t^2+1} - \frac{2}{t^2+1} \right) dt = \frac{t^2}{2} + 2t + \ln(t^2+1) - 2\arctgt + C = \\ &= \frac{2x-3}{2} + 2\sqrt{2x-3} + \ln(2x-2) - 2\arctg\sqrt{2x-3} + C. \end{aligned}$$

b) $\int \frac{dx}{e^x + e^{-x}},$

$$\int \frac{dx}{e^x + e^{-x}} \overset{\boxed{t = e^x, \frac{dt}{dx} = e^x}}{=} \int \frac{dt}{(1+t^{-1})t} = \int \frac{dt}{t^2+1} = \arctgt + C = \arctge^x + C.$$

c) $\int \frac{dx}{3\sin x + \cos x}.$

$$\begin{aligned} \int \frac{dx}{3\sin x + \cos x} &\overset{\boxed{t = \tg \frac{x}{2}, \frac{dt}{dx} = \frac{1+\tg^2 \frac{x}{2}}{2}, \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}}}{=} \int \frac{\frac{2dt}{1+t^2}}{\frac{1}{1+t^2}(6t+1-t^2)} = \\ &= \int \frac{2dt}{-t^2+6t+1} = \int \frac{dt}{t-3-\sqrt{10}} - \int \frac{dt}{t-3+\sqrt{10}} = \ln|t-3-\sqrt{10}| - \ln|t-3+\sqrt{10}| + C = \\ &= \ln \left| \tg \frac{x}{2} - 3 - \sqrt{10} \right| - \ln \left| \tg \frac{x}{2} - 3 + \sqrt{10} \right| + C. \end{aligned}$$