Mathematical analysis 2, WNE, 2018/2019
meeting 26. – solutions

4 June 2019

1. Calculate

$$\int\int_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy.$$

We use substitution $\Phi(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y)$. So $x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$. Moreover,

$$\Phi' = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix},$$

thus

$$\det\Phi' = r(\cos^2\theta + \sin^2\theta) = r.$$

so

$$\int\int_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \int_{(0,\infty)\times(0,2\pi)} |r|e^{r^2}\,dr\,d\theta =$$

$$= \int_0^\infty \int_0^{2\pi} re^{r^2}\,d\theta\,dr = \int_0^\infty \theta\cdot re^{r^2}|_2^{2\pi}\,dr =$$

$$= \pi\int_0^\infty 2re^{r^2}\,dr = \pi\lim_{a\to\infty}\int_0^a 2re^{r^2}\,dr = \pi\lim_{a\to\infty} e^{r^2}|_0^2 = \pi(1-0) = \pi.$$

2. Using the above problem calculate

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

$$\pi = \int\int_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \int\int_{\mathbb{R}^2} e^{-x^2}e^{-y^2}\,dx\,dy =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy = \int_{-\infty}^{\infty} e^{-y^2}\,dy \cdot \int_{-\infty}^{\infty} e^{-x^2}\,dx\,dy = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2,$$

so:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

3. Calculate the surface area between $|y| = \sqrt{x-1}$ and $x-2y-4=0$.

The curves intersect for $x=2$ and $x=10$. But for $x\in[1,2]$ the area is bounded from above by a parabola (symmetric to OX), and for $x\in[2,10]$ it is bounded from above by a parabola and from below by a line. Thus,

$$P = 2\int_0^2 \sqrt{x-1}\,dx + \int_2^{10}\left(\sqrt{x-1} - \left(\frac{x}{2}-2\right)\right)dx = \frac{4(x-1)^{3/2}}{3}\bigg|_1^2 + \frac{2(x-1)^{3/2}}{3}\bigg|_2^{10} - \left(\frac{x^2}{4}-2x\right)\bigg|_2^{10} =$$

$$= \frac{32}{3}.$$

4. In the instant $t = 0\,[s]$ the dancer which moves along OX is at the point 0. He moves with velocity $v = 2t\sin t^2\pi\,[m/s]$. At which point he will be after $1s$?

   The velocity is the derivative of the position, so the position is an integral of the velocity:

$$x = \int_0^1 2t\sin t^2\pi\,dt$$

   we substitute $u = t^2\pi$, $du/dt = 2t\pi$, $u(0) = 0, u(1) = 1$ so:

$$x = \int_0^1 \sin u\pi\,du = -\frac{1}{\pi}\cos u\pi|_0^1 = \frac{2}{\pi}.$$

5. Calculate (using integrals!) the area of a circle of radius 1.

   The circle is parametrized by: $(\cos t, \sin t)$ for $t \in [0, 2\pi]$. Thus:

$$O = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = \int_0^{2\pi} dt = t|_0^{2\pi} = 2\pi.$$

6. Calculate the length of a curve $y = \frac{2x^{3/2}}{3}$, for $0 \leqslant x \leqslant 1$.

$$L = \int_0^1 \sqrt{1 + (t^{1/2})^2}\,dt = \int_0^1 \sqrt{1+t}\,dt$$

   We substitute $u = \sqrt{1+t}$, so $du/dt = \frac{1}{2u}$ and $u(0) = 1, u(1) = \sqrt{2}$

$$L = \int_1^{\sqrt{2}} 2u^2\,du = \left.\frac{2t^3}{3}\right|_1^{\sqrt{2}} = \frac{4\sqrt{2}}{3} - \frac{2}{3}.$$

7. Consider a three-dimensional block with base bounded by $x = 0, y = 0$ and $\sqrt{x} + \sqrt{y} = 1$. The height at point x, y is equal to $h(x, y) = 2x^2y$. Calculate its volume.

   The surface are of a section along y (for given x) is $\int_0^{(1-\sqrt{x})^2} 2x^2y\,dy$, so the volume is:

$$\int_0^1 \left(\int_0^{(1-\sqrt{x})^2} 2x^2y\,dy\right)dx = \int_0^1 x^2y^2|_0^{(1-\sqrt{x})^2}\,dx = \int_0^1 x^2(1-\sqrt{x})^4\,dx =$$

$$\left.\left(-\frac{8x^{\frac{9}{2}}}{9} - \frac{8x^{\frac{7}{2}}}{7} + \frac{x^5}{5} + \frac{3x^4}{2} + \frac{x^3}{3}\right)\right|_0^1 = \frac{1}{630}$$

8. Derive the formula for surface are of a cone of height l and radius of the base r,

   The section at height z has $(l-z)r/l = r(1-z/l)$. So the circumference is $2\pi r(1-z/l)$ and the area is $\pi r^2(1-z/l)^2 = \pi r^2(1 - 2z/l + z^2/l^2)$. Thus, the volume is

$$\int_0^l \pi r^2(1 - 2z/l + z^2/l^2)\,dz = \pi r^2(z - z^2/l + z^3/3l^2)|_0^l = \pi r^2(l - l + l/3) = \frac{\pi l r}{3}.$$

   and the area

$$\int_0^l 2\pi r(1 - z/l)\,dz = 2\pi r(z - z^2/2l)|_0^l = 2\pi r(l - l/2) = \pi r l.$$

9. Calculate the volume of a subset of $\mathbb{R}^3$ bounded by planes $z = 0$ and $z = x$ and the surface of the cylinder $x^2 + y^2 = 4$.

   Section for $x \in [0, 2]$ is a rectangle of height x and base $2\sqrt{4-x^2}$, so the volume is

$$\int_0^2 2x\sqrt{4-x^2}\,dx$$

   We substitute $t = 4 - x^2$, so $dt/dx = -2x$, $t(0) = 4$, $t(2) = 0$, thus

$$= \int_0^4 \sqrt{t}\,dt = \frac{3t^{3/2}}{2}|_0^4 = \frac{3\cdot 8}{2} = 12.$$

10. Using the polar coordinates find the volume of the subset of $\mathbb{R}^3$ bounded by the plane $z = 0$ and the surface of the paraboloid described by $z = 25 - x^2 - y^2$.

Let this fragment of the paraboloid is P. Let $\Phi(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$. Then $\det \Phi' = r$ and $P = \Phi[K]$, where

$$K = \{(r, \theta, z)\colon r \in (0, \sqrt{25 - z}), \theta \in (0, 2\pi), z \in (0, 25)\}.$$

Thus,

$$\int\int\int_P 1\,dx\,dy\,dz = \int\int\int_K r\,d\theta\,dr\,dz = \int_0^{25}\int_0^{\sqrt{25-z}}\int_0^{2\pi} r\,d\theta\,dr\,dz =$$

$$= \int_0^{25}\int_0^{\sqrt{25-z}} 2\pi r\,dr\,dz = \int_0^{25} \pi r^2|_0^{\sqrt{25-z}}\,dz = \int_0^{25}(25 - z)\pi\,dz = (25z - z^2/2)\pi|_0^{25} = \frac{625\pi}{2}.$$