Mathematical Analysis 2 – WNE – Colloquium 2 C – solutions
30 May 2019.

1. Calculate $\nabla f(0,0)$ for the function $f(x,y) = |2x^3\sqrt{y+1}|$, or state that it does not exist.

$$\lim_{h\to 0}\frac{|2\cdot h^3\sqrt{1}| - |2\cdot 0\sqrt{y+1}|}{h} = \lim_{h\to 0} 2|h|h = 0,$$
$$\lim_{h\to 0}\frac{|2\cdot 0\sqrt{1+h}| - |2\cdot 0\sqrt{1}|}{h} = 0,$$

So $f'(0,0) = [0,0]$.

2. Let $f(x,y) = e^{x+y^2}$. Find the maximum of f on the set

$$A = \left\{(x,y)\in\mathbb{R}^2 :\ x^2+y^2 \leqslant 1\right\}.$$

Obviously, there is no extremum in the interior since $\frac{\partial f}{\partial x} = e^{x+y^2} \neq 0$. We check the boundary $F(x,y) = x^2+y^2-1 = 0$. $f' = [e^{x+y^2}, 2ye^{x+y^2}]$, $F' = [2x, 2y]$, so $e^{x+y^2} = \lambda 2x$ and $2ye^{x+y^2} = \lambda 2y$. If $y \neq 0$, then $e^{x+y^2} = \lambda \neq 0$, so $\lambda = \lambda 2x$. Then $x = 1/2$, hence $y = \pm\sqrt{3}/2$. Meanwhile, if $y = 0$, to $x = \pm 1$. We get four points: $(1/2, \pm\sqrt{3}/2)$ and $(\pm 1, 0)$ with values $e^{5/4}$, $e^{5/4}$, e and $1/e$, respectively, so the maximum is $e^{5/4}$, and the minimum is $1/e$.

3. Let $K = \{(x,y)\in\mathbb{R}^2 : 0 \leqslant x \leqslant 1,\ -1/4 \leqslant y \leqslant 0\}$ and let $f(x,y) = \ln(2xy+1)$. Find the maximum and the minimum of f on the set K.
We look for critical points: $f' = (2y/(2xy+1), 2x/(2xy+1)))$, so the only critical point is $(0,0)$ with value 0. We check for the sides and vertices are

- $x = 0$, $f(y) = \ln 1 = 0$, no extrema,
- $x = 1$, $f(y) = \ln(2y+1)$, $f' = 2/(2y+1)$, no extrema,
- $y = -1/4$, $f(x) = \ln(-x/2+1)$, $f' = -/2(-x/2+1)$, no extrema,
- $y = 0$, $f(x) = \ln 1 = 0$, no extrema,
- w $(0,-1/4)$ value 0,
- w $(0,0)$ value 0,
- w $(1,-1/4)$ value $\ln(1/2)$,
- w $(1,0)$ value 0.

So the maximum is 0, and the minimum is $-\ln(1/2)$.

4. Find and classify the critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = x^8 + y^4 - 8xy + 3.$$

$f' = (8x^7 - 8y, 4y^3 - 8x)$, so $y = x^7$ i $x^{21} = 2x$, hence $x = 0$ or $x = \sqrt[20]{2}$ or $x = -\sqrt[20]{2}$. We get three points $(0,0)$, $(\sqrt[20]{2}, 2^{\frac{7}{20}})$, $(-\sqrt[20]{2}, -2^{\frac{7}{20}})$. The matrix of second order derivative is

$$\begin{bmatrix} 56x^6 & -8 \\ -8 & 12y^2 \end{bmatrix},$$

at $(0,0)$ is

$$\begin{bmatrix} 0 & -8 \\ -8 & 0 \end{bmatrix},$$

so the matrix is non-definite. There is no extremum here.
At points $(\sqrt[20]{2}, 2^{\frac{7}{20}})$ and $(-\sqrt[20]{2}, -2^{\frac{7}{20}})$ we get that the matrix is positive definite. We get a local minimum here.
5. Is $f(x,y) = (x^2 + y, 2x - y)$ a local C^1 diffeomorphism of $(-1,1) \times (-1,1)$?

$$f' = \begin{bmatrix} 2x & 1 \\ 2 & -1 \end{bmatrix},$$

so $\det f' = -2x - 2 = -2(x+1) \neq 0$, because $x \in (-1,1)$. Thus, the answer is yes.
6. Is $f(x,y) = (x^2 + y, 2x - y)$ a C^1 diffeomorphism of $(-1,1) \times (-1,1)$?
If $a = x^2 + y$ and $b = 2x - y$, then $x = \pm\sqrt{a+b+1} - 1$, $y = \pm 2\sqrt{a+b+1} - b - 2$, but only with plus we end up in the specified interval. So there exists an inverse function

$$g(a,b) = (\sqrt{a+b+1} - 1, 2\sqrt{a+b+1} - b - 2).$$

7. Let $z(x,y)$ be determined by the equation $\sin(yz^2 + \pi/2) = xz$ and $z(1,0) = 1$. Calculate $\frac{\partial z}{\partial x}(1,0)$ and $\frac{\partial z}{\partial y}(1,0)$.
$F(x,y,z) = \sin(yz^2 + \pi/2) - xz$, so $F' = [-z, z^2\cos(yz + \pi/2), 2zy\cos(yz + \pi/2) - x]$ at point $(1,0,1)$ we get $F' = [-1, 0, -1]$, in particular $-1 \neq 0$, so the implicit function theorem can be applied and

$$z' = (F'_z)^{-1} \cdot F'_{xy} = \frac{1}{-1}[-1, 0] = [1, 0],$$

thus $\frac{\partial z}{\partial x}(1,0) = 1$ i $\frac{\partial z}{\partial y}(1,0) = 0$.
8. Find the equation of the plane tangent to the surface

$$S = \{(x,y,z) \in \mathbb{R}^3 : \sin(yz^2 + \pi/2) = xz\}.$$

at the point $(1,0,1)$.
Since $F'(1,0,1) = [-1,0,-1]$, the equation of the parallel plane going through $(0,0)$ is $-x - z = 0$, so the equation of the plane going through $(1,0,1)$ is $-x - z = -2$.
9. Is the following subset of $\mathbb{R}^3$ a manifold

$$M = \{(x,y,z) \in \mathbb{R}^3 : 2x + 2y = z, x^2 + y^2 = z - 2\}.$$

We get: $F(x,y,z) = (x^2 + y^2 - z + 2, 2x + 2y - z) = (0,0)$, so

$$f' = \begin{bmatrix} 2x & 2y & -1 \\ 2 & 2 & -1 \end{bmatrix}$$

and the rows of this matrix are linearly dependent if $x = y = 1$, so $z = 4$. Which may happen, so it is not a manifold.

10. Decide whether the quadratic form given by the matrix $\begin{bmatrix} -1 & -1 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$ is: positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite.
The determinants are $-1 < 0$, $2 > 0$, $-2 < 0$, so by the Sylvester's criterion it is negative definite (and thus also negative semi-definite).