Mathematical Analysis 2 – WNE – Colloquium 2 A – solutions
30 May 2019.

1. Calculate $\nabla f(0,0)$ for the function $f(x,y) = |2x\ln(1+2y)|$, or state that it does not exist.

$$\lim_{h\to 0} \frac{|2\cdot h\ln(1)| - |2\cdot 0\ln(1)|}{h} = 0,$$
$$\lim_{h\to 0} \frac{|2\cdot 0\ln(1+2h)| - |2\cdot 0\ln(1)|}{h} = 0,$$

So $f'(0,0) = [0,0]$.

2. Let $f(x,y) = x + y^{10}$. Find the maximum of f on the set

$$A = \left\{(x,y) \in \mathbb{R}^2 :\ x^2 + y^10 + 3x \leqslant 1\right\}.$$

Obviously, there is not extremum in the interior since $\frac{\partial f}{\partial x} = 1 \neq 0$. We check the boundary: $F(x,y) = x^2 + y^{10} + 3x - 1 = 0$. $f' = [1, 10y^9]$, $F' = [2x+3, 10y^9]$, so $1 = \lambda(2x+3)$ and $10x^9 = \lambda 10x^9$. If $x \neq 0$, then $\lambda = 1$, thus $x = -1$ and $y^10 = 3$, so $y = \pm\sqrt[10]{3}$. Meanwhile, if $x = 0$, $y^10 = 1$, hence $y = \pm 1$. We get 4 points: $(-1, \pm\sqrt[10]{3})$ i $(0, \pm 1)$. The values in them are 2 and 1 respectively, so maximum is 2.
3. Let $K = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 2\}$ and let $f(x,y) = x^2 - xy^2$. Find the maximum and the minimum of f on the set K.
We look for critical points $f' = (2x - y^2, 2xy)$.We get that $x = 0$ is and only if $y = 0$, so $(0,0)$ and the value 0. Next we check the sides and vertices:

- $x = 0$, $f(y) = 0$, no extrema,
- $x = 2$, $f(y) = 4 - 2y^2$, critical point for $y = 0$ with value 4,
- $y = 0$, $f(x) = x^2$, critical point for $x = 0$ with value 0,
- $y = 2$, $f(x) = x^2 - 4x$, $f' = 2x - 4$, so it has a critical point for for $x = 2$ with value -4,
- at $(0,0)$ the value is 0,
- at $(0,2)$ the value is 0,
- at $(2,0)$ the value is 4,
- at $(2,2)$ the value is -4.

So the maximum is 4 at $(2,0)$, and the minimum is -4 at $(2,2)$
4. Find and classify the critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = x^6 + y^6 - 6x^2y^2 + 3.$$

$f' = (6x^5 - 6y, 6y^5 - 6x)$, so $y = x^5$ i $x^{25} = x$, hence $x = 0$ or $x = 1$ or $x = 1$. We get three critical points $(0,0)$, $(1,1)$, $(-1,-1)$. The matrix of the second order derivative:

$$\begin{bmatrix} 30x^4 & -6 \\ -6 & 30y^4 \end{bmatrix},$$

at $(0,0)$ is

$$\begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix},$$

which is a non-definite matrix. There is no extremum here.
At $(1,1)$ and at $(-1,-1)$ is

$$\begin{bmatrix} 30 & -6 \\ -6 & 30 \end{bmatrix},$$

which is a positive definite matrix (the determinants are $30 > 0$ and $900 - 36 > 0$). So we get a local minimum here.

5. Is $f(x,y) = (x^2y, x)$ a local C^1 diffeomorphism of $(0,\infty) \times (-\infty,\infty)$?

$$f' = \begin{bmatrix} 2xy & x^2 \\ 1 & 0 \end{bmatrix},$$

so $\det f' = -x^2 \neq 0$ for $(x,y) \in (0,\infty) \times (-\infty,\infty)$. Thus, it is.

6. Is $f(x,y) = (x^2y, x)$ a C^1 diffeomorphism of $(0,\infty) \times (-\infty,\infty)$?
Yes, there is a reverse function $g(a,b) = (b, a/b^2)$, which is also of C^1 class on the set of values of f, i.e. $(-\infty,\infty) \times (0,\infty)$.

7. Let $z(x,y)$ be determined by the equation $e^{yz^2} = xz$ and $z(1,0) = 1$. Calculate $\frac{\partial z}{\partial x}(1,0)$ and $\frac{\partial z}{\partial y}(1,0)$.
$F(x,y,z) = e^{yz^2} - xz$, so $F' = [-z, z^2e^{yz^2}, 2zye^{yz^2} - x]$ at $(1,0,1)$ we get $F' = [-1,1,-1]$, in particular $2zye^{yz^2} - x = 0 - 1 - 1 \neq 0$, so we can apply the implicit function theorem, and

$$z' = (F'_z)^{-1} \cdot F'_{xy} = \frac{1}{-1}[-1,1] = [1,-1],$$

so $\frac{\partial z}{\partial x}(1,0) = 1$ i $\frac{\partial z}{\partial y}(1,0) = -1$.

8. Find the equation of the plane tangent to the surface

$$S = \{(x,y,z) \in \mathbb{R}^3 : \exp(yz^2) = xz\}.$$

at the point $(1,0,1)$.
Since $F'(1,0,1) = [-1,1,-1]$, the equation of the linear plane which is parallel is $-x + y - z = 0$, so the equation of the plane going through $(1,0,1)$ is $-x + y - z = -2$.

9. Is the following subset of $\mathbb{R}^3$ a manifold

$$M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + 2z^2 = 1, \ \ 2z = x^2 + y^2\}.$$

We get: $F(x,y,z) = (x^2 + y^2 + 2z^2 - 1, x^2 + y^2 - 2z) = (0,0)$, so

$$f' = \begin{bmatrix} 2x & 2y & 4z \\ 2z & 2y & -2 \end{bmatrix}$$

and the rows are linearly dependent only if $4z = -2$, i.e. $z = -1/2$. But then we get $x^2 + y^2 = 1/2$ and $x^2 + y^2 = -1$, which is a contradiction, so they are always linearly independent, and so it is a manifold.

10. Decide whether the quadratic form given by the matrix $\begin{bmatrix} -2 & 0 & -1 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$ is: positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite.
The determinants are $-2 < 0$, $2 > 0$, $-2 < 0$, so by the Sylvester's criterion it is negative definite (and thus also negative semi-definite).