Mathematical analysis 2, WNE, 2018/2019
meeting 25. – short test – solutions

28 May 2019

A

1. Find the minimum and maximal values of $f(x, y) = 2x + y^2$ on the set $\{(x, y) \in \mathbb{R}^2 : 4x^2 + y^2 = 4\}$.
   $f(x, y) = (2, 2y)$, $F(x, y) = 4x^2 + y^2 - 4 = 0$, $F'(x, y) = (8x, 2y)$, so $2 = 8\lambda x$ and $2y = 2\lambda y$. When $y \neq 0$, then $\lambda = 1$, hence $x = 1/4$ and then
   $$y^2 = 4 - 4/16 = 15/4,$$
   so $y = \pm\sqrt{15}/2$. If $y = 0$, then $x = \pm 1$. We get 4 points: $(1/4, \pm\sqrt{15}/2)$ and $(\pm 1, 0)$ and the value is $1/2 + 15/4 = 17/4$, $17/4$, 2, -2 respectively. Maximum is $17/4$, and minimum is -2.
2. Calculate $\int\int_D 2x^2 y\, dxdy$ on the set D bounded by curves $y = -x^2 + 1$ and $y = 0$.
   $$\int\int_D xy^2\, dxdy = \int_{-1}^{1}\int_{0}^{1-x^2} 2x^2 y\, dy\, dx = \int_{-1}^{1} x^2y^2|_0^{1-x^2}\, dx =$$
   $$= \int_{-1}^{1} (x^2 - x^4)\, dx = (x^3/3 - x^5/5)|_{-1}^{1} = 2/15 + 2/15 = 4/15.$$

B

1. Find the minimum and maximal values of $f(x, y) = x^2 + 2y$ on the set $\{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 = 4\}$.
   $f(x, y) = (2x, 2)$, $F(x, y) = x^2 + 4y^2 - 4 = 0$, $F'(x, y) = (2x, 8y)$, thus $2 = 8\lambda y$ and $2x = 2\lambda x$. If $x \neq 0$, then $\lambda = 1$, hence $y = 1/4$ and then
   $$x^2 = 4 - 4/16 = 15/4,$$
   thus $x = \pm\sqrt{15}/2$. If $x = 0$, to $y = \pm 1$. We get 4 points: $(\pm\sqrt{15}/2, 1/4)$ and $(0, \pm 1)$ and the values are $1/2 + 15/4 = 17/4$, $17/4$, 2, -2 respectively. Maximum is $17/4$, and minimum is -2.
2. Calculate $\int\int_D 2x^2 y\, dxdy$ on the set D bounded by curves $y = x^2 - 1$ and $y = 0$.
   $$\int\int_D xy^2\, dxdy = \int_{-1}^{1}\int_{x^2-1}^{0} 2x^2 y\, dy\, dx = \int_{-1}^{1} x^2y^2|_{x^2-1}^{0}\, dx =$$
   $$= \int_{-1}^{1} (x^4 - x^2)\, dx = (x^5/5 - x^3/3)|_{-1}^{1} = -2/15 - 2/15 = -4/15.$$

C

1. Find the minimum and maximal values of $f(x, y) = x^2 + 2y + 2z$ on the set $\{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 + z^2 = 5\}$.
   $f(x, y, z) = (2x, 2, 2)$, $F(x, y) = x^2 + 4y^2 + z^2 - 4 = 0$, $F'(x, y) = (2x, 8y, 2z)$, so $2 = 8\lambda y$, $2 = 2\lambda z$ and $2x = 2\lambda x$. If $x \neq 0$, then $\lambda = 1$, hence $y = 1/4$ and $z = 1$ and then
   $$x^2 = 5 - 4/16 - 1 = 15/4,$$
   hence $x = \pm\sqrt{15}/2$. If $x = 0$, then $\lambda \neq 0$ i $y = 1/(4\lambda)$, $z = 1/\lambda$, so $1/\lambda^2 + 4/\lambda^2 = 20$, thus $1/\lambda^2 = 5$, $\lambda = \pm 1/2$, so $y = \pm 1/2$, $z = \pm 2$. We get 4 points: $(\pm\sqrt{15}/2, 1/4, 1)$ and $(0, \pm 1/2, \pm 2)$ and the values are $1/2 + 15/4 + 2 = 25/4$, $25/4$, 5, -5 respectively. Maximum is $25/4$, and minimum is -5.

2. Calculate $\int\int_D \frac{2y}{(x^2+1)^2}\,dxdy$ on the set D bounded by curves $y = x^4 - 1$ and $y = 0$.

$$\int\int_D \frac{2y}{(x^2+1)^2}\,dxdy = \int_{-1}^{1}\int_{x^4-1}^{0} \frac{2y}{(x^2+1)^2}\,dy\,dx = \int_{-1}^{1} \frac{y^2}{(x^2+1)^2}|_{x^4-1}^{0}\,dx =$$

$$= \int_{-1}^{1} -(x^2-1)^2\,dx = -(x^5/5 - 2x^3/3 + x)|_{-1}^{1} = -((1/5 - 1/3 + 1) - (-1/5 + 1/3 - 1)) = -32/15.$$

D

1. Find the minimum and maximal values of $f(x, y) = 2x + y^2 + 2z$ on the set $\{(x, y) \in \mathbb{R}^2 \colon 4x^2 + y^2 + z^2 = 5\}$.
   $f(x, y, z) = (2, 2y, 2)$, $F(x, y) = 4x^2 + y^2 + z^2 - 4 = 0$, $F'(x, y) = (8x, 2y, 2z)$, thus $2 = 8\lambda x$, $2 = 2\lambda z$ and $2y = 2\lambda y$. If $y \neq 0$, to $\lambda = 1$, so $x = 1/4$ and $z = 1$ and then

   $$y^2 = 5 - 4/16 - 1 = 15/4,$$

   so $y = \pm\sqrt{15}/2$. If $y = 0$, then $\lambda \neq 0$ and $x = 1/(4\lambda)$, $z = 1/\lambda$, so $1/\lambda^2 + 4/\lambda^2 = 20$, thus $1/\lambda^2 = 5$, $\lambda = \pm 1/2$, so $x = \pm 1/2$, $z = \pm 2$. We get 4 points: $(1/4, \pm\sqrt{15}/2, 1)$ and $(\pm 1/2, 0, \pm 2)$ and values $1/2 + 15/4 + 2 = 25/4$, $25/4$, 5, -5 respectively. Maximum is $25/4$, and minimum is -5.

2. Calculate $\int\int_D \frac{2y}{(x^2+1)^2}\,dxdy$ on the set D bounded by curves $y = 1 - x^4$ and $y = 0$.

$$\int\int_D \frac{2y}{(x^2+1)^2}\,dxdy = \int_{-1}^{1}\int_{0}^{1-x^4} \frac{2y}{(x^2+1)^2}\,dy\,dx = \int_{-1}^{1} \frac{y^2}{(x^2+1)^2}|_{0}^{1-x^4}\,dx =$$

$$= \int_{-1}^{1} (1-x^2)^2\,dx = (x^5/5 - 2x^3/3 + x)|_{-1}^{1} = (1/5 - 1/3 + 1) - (-1/5 + 1/3 - 1) = 32/15.$$