Mathematical analysis 2, WNE, 2018/2019
meeting 25. – solutions

28 May 2019

1. Is there a function of C^2 class, $f\colon \mathbb{R}^2 \to \mathbb{R}$, such that

$$\frac{\partial f}{\partial x} = x\sin y,$$

$$\frac{\partial f}{\partial y} = y\cos x?$$

If such a function exists, then

$$\frac{\partial^2 f}{\partial x \partial y} = x\cos y,$$

$$\frac{\partial^2 f}{\partial y \partial x} = -y\sin x$$

but these should be equal. So such a function does not exist.

2. Let $f(x,y) = x + y^2$. Find the maximum of f on the set

$$A = \left\{(x,y) \in \mathbb{R}^2 :\ 3x^2 + 2y^2 + 8x \leqslant 1\right\}.$$

It is clear that f has no critical points because $\partial f \partial x = 1 \neq 0$. We check its maxima on the boundary, i.e. $F(x,y) = 3x^2 + 2y^2 + 8x - 1 = 0$. $f' = (1, 2y)$ and $F'(x,y) = (6x+8, 4y)$, so $1 = \lambda(6x+8)$ i $2y = \lambda 4y$. If $y \neq 0$, to $\lambda = 1/2$, then $1 = 3x + 4$, $x = -1$, $3 + 2y^2 - 8 = 1$, so $y^2 = 3$, $y = \pm\sqrt{3}$. When $y = 0$, to $3x^2 + 8x - 1 = 0$. $\Delta = 64 + 12 = 76$. Thus, $x = (3 \pm \sqrt{19})/3$. So we have four candidates $(-1, \pm\sqrt{3})$ and $(3 \pm \sqrt{19})/3, 0)$. The values are $2, 2, 3 + \sqrt{19})/3$ and $3 - \sqrt{19})/3$, respectively, and the value $3 + \sqrt{19})/3$ is the maximum.

3. Let $K = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 2\}$ and let $f(x,y) = x^2 + y - xy^2$. Find the maximum and the minimum of f on the set K.

We find critical points. $f' = (2x - y^2, 1 - 2xy)$, so in critical points we have $x = y^2/2$, hence $y^3 = 1$, so $y = 1$ and $x = 1/2$. At this point $f(x,y) = 3/4 - 1/2 = 1/4$. We look for extrema on the sides and in the vertices:

- for $x = 0$ we get $f(y) = y$, so there are no critical points,
- for $x = 1$ we get $f(y) = -y^2 + y + 1$, $f' = -2y + 1$, has a critical point for $y = 1/2$ and the value is $5/4$,
- for $y = 0$ we get $f(x) = x^2$ has critical point for $x = 0$ and the value is 0,
- for $y = 2$ we get $f(x) = x^2 - 4x + 2$, $f' = 2x - 4$, has a critical point outside of $(0,1)$,
- at $(0,0)$ the value is 0,
- at $(0,2)$ the value is 2,
- at $(1,0)$ the value is 1,
- at $(1,2)$ the value is -1.

So the minimum is -1 at $(1,2)$, and maximum 2 at $(0,2)$.

4. Find and classify the critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = x^3 + y^3 + 3xy + 3.$$

$f' = (3x^2 + 3y, 3y^2 + 3x)$, so the critical points are $(-1, -1)$ and $(0, 0)$. The matrix of the second order derivative is

$$\begin{bmatrix} 6x & 3 \\ 3 & 6y \end{bmatrix},$$

at $(-1, -1)$ is

$$\begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix},$$

is negative definite (the determinants are -6 and 27) – we have a local maximum, at $(0, 0)$ we get

$$\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix},$$

is non-definite, so there is no extremum here.

5. Let $f(r, \theta) = \big(r^2 \cos(2\theta), r^2 \sin(2\theta)\big)$ where $(r, \theta) \in (0, 1) \times (0, 2\pi)$. Sketch the image of f.

   The point $f(r, \theta)$ is at distance of r^2 from $(0, 0)$ with angle 2θ from x-axis, so the image is the full circle $(0, 0)$ of radius 1, without boundary and its centre.

6. Let f be the function from the previous problem.

   a) Is f a local diffeomorphism of the set $(0, 1) \times (0, 2\pi)$?

   $$f' = \begin{bmatrix} 2r\cos(2\theta) & -2r^2\sin(2\theta) \\ 2r\sin(2\theta) & 2r^2\cos(2\theta) \end{bmatrix},$$

   has determinant

   $$2r^3(\cos^2(2\theta) + \sin^2(2\theta)) = 3r^3 \neq 0$$

   for $r \in (0, 1)$, So by the inverse function theorem, locally there exists an inverse function of class C^1, so it is locally a diffeomorphism.

   b) Is f a diffeomorphism of the set $(0, 1) \times (0, 2\pi)$?
   No, because it is not one-to-one, $f(1/2, \pi/2) = (-1/4, 0) = f(1/2, 3\pi/2)$.

7. Let $z(x, y)$ be determined by the equation $\sin(xz) = yz$ and $z(1, 0) = 0$. Calculate $\frac{\partial z}{\partial x}(1, 0)$ and $\frac{\partial z}{\partial y}(1, 0)$.

   So $F(x, y, z) = \sin(xz) - yz$, $F'(x, y, z) = [z\cos xz, -z, x\cos xz - y]$. At $(1, 0, 0)$ we get $[0, 0, 1]$, and $1 \neq 0$, so we can apply implicit function theorem

   $$z' = (F'_z)^{-1} \cdot F'_{xy} = \frac{1}{1} \cdot [0, 0] = [0, 0],$$

   hence,

   $$\frac{\partial z}{\partial x}(1, 0) = 0 = \frac{\partial z}{\partial y}(1, 0).$$

8. Find the equation of the plane tangent to the surface

   $$S = \{(x, y, z) \in \mathbb{R}^3 : \sin(xz) = yz\}.$$

   at the point $(1, 0, 0)$.

   $F'(x, y, z) = [z\cos xz, -z, x\cos xz - y]$, at $(1, 0, 0)$ is $[0, 0, 1]$, so the equation of the tangent linear space is $z = 0$, also going through this point is $z = 0$.

9. Is the following subset of $\mathbb{R}^3$ a manifold

   $$M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, \;\; z = x^2 + y^2\}.$$

$F(x, y, z) = (x^2 + y^2 + z^2 - 1, x^2 + y^2 - 1)$, so

$$F'(x, y, z) = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & 1 \end{bmatrix},$$

these rows are linearly independent if $2z \neq 1$, i.e. for $z \neq 1/2$. For $z = 1/2$ we get $x^2 + y^2 = 3/4$ and $x^2 + y^2 = 1/2$ – which is a contradiction, so $z = 1/2$ which dos not hold in the considered set. So the rows of F' are linearly independent. It is a manifold.

10. Decide whether the quadratic form given by the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is: positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite.

    The subsequent determinants are $1 > 0$, $2 > 0$ and $2 > 0$, so by Sylvester's criterion it is positive definite (thus also positive semidefinite).