Mathematical analysis 2, WNE, 2018/2019
meeting 24. – solutions

23 May 2019

1. Calculate

a) $\int_2^3 \left(\int_0^1 6xy\,dx\right)\,dy$,

$$\int_2^3 \left(\int_0^1 6xy\,dx\right)\,dy = \int_2^3 3x^2y|_0^1\,dy = \int_2^3 3y\,dy = 3y^2/2|_2^3 = 27/2 - 12/2 = 15/2.$$

b) $\int_1^3 \left(\int_{-2}^1 (4x^3 + 6xy^2)\,dy\right)\,dx$,

$$\int_1^3 \left(\int_{-2}^1 (4x^3 + 6xy^2)\,dy\right)\,dx = \int_1^3 (4x^3y + 2xy^3)|_{-2}^1\,dx =$$

$$= \int_1^3 (4x^3 + 2x + 8x^3 + 16x)\,dy = (3x^4 + 9x^2)|_1^3 = 312.$$

c) $\int_0^1 \int_0^\pi e^x \sin y\,dy\,dx$,

$$\int_0^1 \int_0^\pi e^x \sin y\,dy\,dx = \int_0^1 -e^x \cos y|_0^\pi\,dx = \int_0^1 2e^x\,dx = 2e^x|_0^1 = 2e - 2.$$

d) $\lim_{n\to\infty} \int_0^1 \int_0^1 x^n y^n\,dx\,dy$,

$$\lim_{n\to\infty} \int_0^1 \int_0^1 x^n y^n\,dx\,dy = \lim_{n\to\infty} \int_0^1 x^n y^n/(n+1)|_0^1\,dy =$$

$$= \lim_{n\to\infty} \int_0^1 y^n/(n+1)\,dy = lim_{n\to\infty} y^{n+1}/(n+1)^2|_0^1 = \lim_{n\to\infty} 1/(n+1)^2 = 0.$$

e) $\int\int_D xy^2\,dx\,dy$, where D is a subset of $\mathbb{R}^2$ between curves $y = x^2$ and $y = x^3$.

$$\int\int_D xy^2\,dx\,dy = \int_0^1 \int_{x^2}^{x^3} xy^2\,dy\,dx = \int_0^1 xy^3/3|_{x^2}^{x^3}\,dx =$$

$$= \frac{1}{3}\int_0^1 x^{10} - x^7\,dx = \frac{1}{3}(x^{11}/11 - x^9/9)|_0^1 = \frac{1}{3}(1/11 - 1/9) = 2/297.$$

f) $\int\int_D (6x + 2y^2)\,dx\,dy$, where D is a subset of $\mathbb{R}^2$ between curve $y = x^2$ and line $x + y = 2$.

$$\int\int_D (6x + 2y^2)\,dx\,dy = \int_{-2}^1 \int_{x^2}^{2-x} (6x + 2y^2)\,dy\,dx = \int_{-2}^1 (6xy + 2y^3/3)|_{x^2}^{2-x}\,dx =$$

$$= \int_{-2}^1 (-2x^6/3 - 20x^3/3 - 2x^2 + 4x + 16/3)\,dx = (-2x^7/21 - 5x^4/3 - 2x^3/3 + 2x^2 + 16x/3)|_{-2}^1 = 117/7.$$

g) $\int_0^1 \int_0^{x^3} e^{y/x}\,dy\,dx$,

$$\int_0^1 \int_0^{x^3} e^{y/x}\,dy\,dx = \int_0^1 xe^{y/x}|_0^{x^3}\,dx = \int_0^1 \left(xe^{x^2} - x\right)\,dx = (e^x(x-1) - x^2/2)_0^1 = -1/2 + 1 = 1/2.$$

2. Knowing that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

find

a) $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx$,

We put $x = t\sqrt{2}$, so $dt/dx = 1/\sqrt{2}$, and also $t \to \pm\infty$, when $x \to \pm\infty$, thus

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \frac{\sqrt{2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}\,dt = 1.$$

b) $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-\frac{x^2}{2}}\,dx$,

$$= 0,$$

because the function is odd.

c) $\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}}\,dx$, where $\sigma > 0$.

$$\int_{-a}^{a} x \cdot x \cdot e^{-\frac{x^2}{2\sigma^2}}\,dx = x^3 e^{-\frac{x^2}{2\sigma^2}} |_{-a}^{a} + \sigma^2 \int_{-a}^{a} e^{-\frac{x^2}{2\sigma^2}} = \sigma^2 \int_{-a}^{a} e^{-\frac{x^2}{2\sigma^2}}\,dx$$

Thus,

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}}\,dx = \frac{1}{\sigma\sqrt{2\pi}} \sigma^2 \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}\,dx = \frac{\sigma}{\sqrt{2\pi}} \sqrt{2\sigma^2\pi} = \sigma^2.$$