Mathematical analysis 2, WNE, 2018/2019
meeting 23. – solutions

21 May 2019

1. Show that among rectangles with vertices in the circle of radius 1, the square maximises the area.

   We take (x, y) as the vertex of the rectangle, which has to be on the circle, so $F(x, y) = x^2 + y^2 - 1 = 0$ and we maximize the surface area $f(x, y) = 2x \cdot 2y = 4xy$. Thus, $f'(x, y) = (2y, 2x)$ and $F'(x, y) = (2x, 2y)$, so $2y = \lambda 2x$ and $2x = \lambda 2y$. Thus, $x = \lambda^2 = y$, so $x = y = \pm\sqrt{2}/2$. But such a point is exactly one of the vertices of the square.

2. If we inscribe rectangles in the remaining pieces of the circle so that one of the sides is contained in edge of the square, what dimensions should we choose so that the rectangles maximise area?

   We take (x, y) as a point on the circle, so $F(x, y) = x^2 + y^2 - 1 = 0$ (but we are interested only in $\sqrt{2}/2 \leqslant x \leqslant 1$) and the area is $f(x, y) = (x - \sqrt{2}/2)2y = 2xy - y\sqrt{2}$. Thus, $f'(x, y) = (2y, 2x - \sqrt{2})$ and $F'(x, y) = (2x, 2y)$, so $2y = \lambda 2x$, ale $2x - \sqrt{2} = \lambda 2y$, hence $x - \sqrt{2}/2 = \lambda^2 x$, so $x = \frac{\sqrt{2}/2}{1-\lambda^2}$, if $\lambda \neq \pm 1$ (otherwise, we get a contradiction). And then, $y = \frac{\lambda\sqrt{2}/2}{1-\lambda^2}$, therefore $1/2 + \lambda^2/2 = 1 - \lambda^2$, so $3\lambda^2 = 1$, thus $\lambda = \pm 1/\sqrt{3}$, so

$$x = \frac{\sqrt{6}}{2(\sqrt{3} - 1)}$$
$$y = \frac{\sqrt{6}}{2(3 - \sqrt{3})}$$

   is the vertex determining the rectangle.

3. Among the points which belong to the intersection of the plane $x + 2y + 3z = 3$ and the cone $z^2 = x^2 + y^2$ find those closest and farthest from the origin.

   $F(x, y, z) = (x + 2y + 3z - 3, z^2 - x^2 - y^2) = (0, 0)$ and $f(x, y, z) = x^2 + y^2 + z^2$. So $f' = (2x, 2y, 2z)$ and $F_1' = (1, 2, 3)$, $F_2' = (-2x, -2y, 2z)$, thus

$$\begin{cases} 2x = \lambda_1 - 2\lambda_2 x \\ 2y = 2\lambda_1 - 2\lambda_2 y \\ 2z = 3\lambda_1 + 2\lambda_2 z \\ x + 2y + 3z = 3 \\ z^2 = x^2 + y^2 \end{cases}$$

   The first two equations imply $y = 2x$. So $z^2 = 5x^2$ and $3x + 3z = 3$, hence $x = 1 - z$, so $0 = 4z^2 - 10z + 5$, therefore $z = (5 \pm \sqrt{5})/4$, $x = (-1 \mp \sqrt{5})/4$ and $y = (-1 \mp \sqrt{5})/2$. So point $(-1 - \sqrt{5})/4, (-1 - \sqrt{5})/2, (5 + \sqrt{5})/4))$ is the furthest and $(-1 + \sqrt{5})/4, (-1 + \sqrt{5})/2, (5 - \sqrt{5})/4))$ is the nearest.

4. Determine the maximum and minimum values of the function $f(x, y)$ on set $S \subseteq \mathbb{R}^2$, where:

   a) $f(x, y) = x^2y^2$, $S = \{(x, y) \in \mathbb{R}^2 \colon x^2 + 4y^2 = 4\}$,
   $f' = (2xy^2, 2x^2y)$, $F' = (2x, 8y)$, so $2xy^2 = 2\lambda x$, $2x^2y = 8\lambda y$, which if $x \neq 0$ implies $y^2 = \lambda$ and $x^2 = 4\lambda$, hence $5\lambda = 4$, so $\lambda = 4/5$. The values of f at these points is $64/25$ and is maximal. If $x = 0$, then we get value zero 0, which is minimal.

   b) $f(x, y) = x^2 + y^2$, $S = \{(x, y) \in \mathbb{R}^2 \colon 2x + 2y = 6\}$.
   Since S is a plane, f does not have maximal value. $f' = (2x, 2y)$ $F' = (2, 2)$, so $x = y = 3\sqrt{2}/2$ and the value is 9.

5. Determine the maximum and minimum values of the function $f(x, y, z)$ on set $S \subseteq \mathbb{R}^3$, where:

a) $f(x, y, z) = 3x + 2y + z$, $S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$,
$f'(x, y, z) = (3, 2, 1)$, $F(x, y, z) = (2x, 2y, 2z)$, so $3 = 2\lambda x$, $2 = 2\lambda y$ i $1 = 2\lambda z$, hence $x = 3/2\lambda$, $y = 1/\lambda$, $z = 1/2\lambda$, so $\frac{14}{4\lambda^2} = 1$, thus $\lambda = \pm\sqrt{7/2}$ and since $f(x, y, z) = \frac{7}{\lambda}$, the minimal value is $-\sqrt{14}$, and the maximal is $\sqrt{14}$.

b) $f(x, y, z) = x^2 + y^2 + z^2$, $S = \{(x, y, z) \in \mathbb{R}^3 \colon 3x + 2y + z = 6\}$,
Again, it is clear that the maximal value does not exist. To find the minimal value we get $f' = (2x, 2y, 2z)$ $F' = (3, 2, 1)$, so $x = 3\lambda/2$ $y = \lambda$, $z = \lambda/2$ and $14\lambda = 6$, thus $\lambda = 3/7$. So the minimal value is $63/98$.

c) $f(x, y, z) = xyz$, $S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 \leqslant 1, x + y + z = 1\}$,
Ignoring the first condition we see that $f' = (yz, xz, xy)$ and $F'(x, y, z) = (1, 1, 1)$. So $yz = xz = xy$, $x + y + z = 1$. Hence we have points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ and $(1/3, 1/3, 1/3)$. All do satisfy the inequality and give values 0 and $1/27$. If we additionally consider $F_2 = x^2 + y^2 + z^2 - 1 = 0$, then $F_2' = (2x, 2y, 2z)$, so $yz = \lambda_1 + 2\lambda_2 x$, $xz = \lambda_1 + 2\lambda_2 y$, $xy = \lambda_1 + 2\lambda_2 z$, then we also get solutions $(-1/3, 2/3, 2/3)$, $(2/3, 2/3, -1/3)$, $(2/3, 2/3, -1/3)$. The value in theses points is $-2/27$. So the minimal value is $-2/27$, and the maximal values is $1/27$.

6. Using the Kuhn-Tucker theorem, find the maximum value of the function $f(x, y) = x + ay$ on the set

$$M = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leqslant 1, x + y \geqslant 0\}.$$

So we minimize $-x - ay$ while $x^2 + y^2 - 1 \leqslant 0$, $-x - y \leqslant 0$. So $L = -x - ay + \lambda_1(x^2 + y^2 - 1) - {}_2(x + y)$ and $-1 + 2\lambda_1 x - \lambda_2 = 0$ and $-a + 2\lambda_1 y - \lambda_2 = 0$ and $\lambda_1(x^2 + y^2 - 1) = 0$ and $\lambda_2(x + y) = 0$. We get the following solutions (remember that $\lambda_1, \lambda_2 \geqslant 0$):

(i) for $a \leqslant -1$, $(1/\sqrt{2}, -1/\sqrt{2})$, $(-1/\sqrt{a^2 + 1}, -a/\sqrt{a^2 + 1})$ with values $(-1 - a)/\sqrt{2}$ and 1 – the maximum of the original function is $(1 + a)/\sqrt{2}$,

(ii) for $a \geqslant -1$, $(-1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{a^2 + 1}, a/\sqrt{a^2 + 1})$ with values $(1 + a)/\sqrt{2}$ and -1 – the maximum of the original function is 1.