Mathematical analysis 2, WNE, 2018/2019
meeting 22. – solutions

16 May 2019

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function of C^1 class such that for some $0 < k < 1$ and all $x \in \mathbb{R}$, $|f'(x)| \leqslant k$. Prove that $y = x + f(x)$ is a diffeomeorphism.

   Obviously, locally at every point there exists an inverse function of C^1 class, since $y'(x) = 1 + f'(x) \neq 0$. Moreover, $y(x)$ is one-to-one. Indeed, if $y(x_1) = y(x_2)$ and $x_1 < x_2$, then there exists $c \in (x_1, x_2)$, such that $y'(c) = 0$, which is impossible. Thus, there exists an inverse function on the whole line and it has to coincide with the local functions of C^1 class so it is also of C^1 class.

2. Let
$$E = \{(x, y) \in \mathbb{R}^2 \colon x^2 - 2xy + 2y^2 = 1\}.$$
   Use the Lagrange multipliers method to find points of E which are closest to and farthest from the origin of the coordinate system.

   The calculations in these problem are quite nasty! So we consider $d(x, y) = x^2 + y^2$, $d'(x, y) = [2x, 2y]$ on $F(x, y) = x^2 - 2xy + 2y^2 - 1 = 0$, so $F'(x, y) = [2x - 2y, -2x + 4y]$. Thus we are looking for points of E such that
$$\begin{cases} 2x = \lambda(2x - 2y) \\ 2y = \lambda(-2x + 4y) \end{cases}.$$
   If $\lambda = 0$, then $x = y = 0$, but this in not in E, so $\lambda \neq 0$. Therefore, $y = (\lambda - 1)x/\lambda$, so $(\lambda^2 - 3\lambda + 1)x = 0$, but if $x = 0$, we get $y = 0$, so $x \neq 0$ and then $\lambda^2 - 3\lambda + 1$, thus $\lambda = \frac{3 \pm \sqrt{5}}{2}$, but then we get that $y = \frac{(\pm\sqrt{5}-1)x}{2}$ and putting it into $x^2 - 2xy + 2y^2 - 1 = 0$ one can calculate that
$$x = \pm\frac{1}{5 \pm 2\sqrt{5}}.$$
   For these four points the least value of $d(x, y)$ is achieved at
$$x = \pm\frac{1}{5 + 2\sqrt{5}}$$
   and then
$$y = \pm\frac{-15 - 7\sqrt{5}}{10}.$$

3. Use the Lagrange multipliers method to find all those points on the ellipse $x^2 + 2y^2 = 1$, which are nearest to and furthest from the line $x + y = 2$.

   We can measure the distance from this line using $d(x, y) = (x + y - 2)^2$. Then $d'(x, y) = [2x + 2y - 2, 2x + 2y - 2]$ and $F = x^2 + 2y^2 - 1 = 0$, so $F'(x, y) = [2x, 4y]$. So we are looking for points such that
$$\begin{cases} 2x = \lambda(2x + 2y - 2) \\ 4y = \lambda(2x + 2y - 2) \end{cases},$$
   In particular, $x = 2y$, and then $4y^2 + 2y^2 = 1$, so $y = \pm 1/\sqrt{6}$, but $x = \pm 2/\sqrt{6}$, which gives values of d: $(11 - \sqrt{6})/2$ and $(11 + \sqrt{6})/2$ respectively, and is smaller in the first case, i.e. for $(2/\sqrt{6}, 1/\sqrt{6})$ - it is the nearest point and $-(2/\sqrt{6}, -1/\sqrt{6})$ is the furthest.

4. Find supremum and infimum of $f(x,y,z) = x^2 - yz$ on the sphere $x^2 + y^2 + z^2 = 1$.

   Then $F(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$ and $g(x,y,z) = x^2 - yz$. We get $F'(x,y,z) = [2x, 2y, 2z]$ and $g'(x,y,z) = [2x, -z, -y]$. We look for $\lambda \in \mathbb{R}$, such that $[2x, -z, -y] = \lambda[2x, 2y, 2z]$, i.e.

$$\begin{cases} 2x = 2\lambda x \\ -z = 2\lambda y \\ -y = 2\lambda z \end{cases}$$

   If $x \neq 0$ then $\lambda = 1$, thus $-y = -4y$, so $y = 0 = z$, and $x = \pm 1$. So we get two points in which we may have extrema: $(1,0,0)$ and $(-1,0,0)$ (and $\lambda = 1$). The value at these points is 1. On the other hand, if $x = 0$, then $z = \lambda^2 z$ and $y = 4\lambda^2 y$. Both cannot be equal to zero, so $\lambda = \pm 1/2$ and $y = \pm z$, but since $x^2 + y^2 + z^2 - 1 = 0$, we get $2y^2 = 2z^2 = 1$. Thus, for $\lambda = 1/2$ we have points $(0, \sqrt{2}/2, -\sqrt{2}/2)$ and $(0, -\sqrt{2}/2, \sqrt{2}/2)$, and for $\lambda = -1/2$ we have $(0, \sqrt{2}/2, \sqrt{2}/2)$ and $(0, \sqrt{2}/2, \sqrt{2}/2)$. In these point the value of g is respectively $-1/2, -1/2, 1/2$ and $1/2$. So the minimal value is $-1/2$ and maximal is 1.

5. Find the minimal value of $f(x,y,z) = x + y + z$ on the sphere $x^2 + y^2 + z^2 = a^2$.

   $f'(x,y,z) = [1,1,1]$ and $F' = [2x, 2y, 2z]$. We may assume that $a > 0$, and then $(0,0,0)$ is not on the sphere and $x = y = z$, so $x = y = z = \pm\frac{a}{\sqrt{3}}$. Thus, the maximal value is $3\frac{a}{\sqrt{3}}$.

6. Prove the following inequality between the arithmetic and square mean, i.e.

$$\frac{x+y+z}{3} \leqslant \sqrt{\frac{x^2+y^2+z^2}{3}},$$

   for $x, y, z \geqslant 0$.

   It suffices to notice (on the basis of the previous problem) that if $x^2 + y^2 + z^2 = a^2$, we have

$$\frac{x+y+z}{3} \leqslant \frac{a}{\sqrt{3}} = \sqrt{\frac{a^2}{3}} = \sqrt{\frac{x^2+y^2+z^2}{3}}.$$