Mathematical analysis 2, WNE, 2018/2019
meeting 21. – solutions

14 May 2019

1. Check the theorem of implicit function for

$$\begin{cases} x + y_1 y_2^2 = 0 \\ x + y_1 = 0 \end{cases},$$

and points $(-1,1,1)$ and $(0,0,1)$.

It corresponds to $F(x,y_1,y_2) = (0,0)$, where $F(x,y_1,y_2) = (x + y_1 y_2^2, x + y_1)$. The question is whether we can determine $y_, y_2$ from x in a neighbourhood of this point. Let us consider neighbourhood of $(-1,1,1)$ and try to solve the problem manually. We have that $y_1 = -x$. We also know that $y_1 \neq 0$ (neighbourhood of 1) so $y_2^2 = -x/y_1$. We know that y_2 is positive (neighbourhood of 1), thus $y_2 = \sqrt{-x/y_1} = \sqrt{1} = 1$. Then $H(x) = (-x, 1)$.

We check whether it is consistent with the theorem. We have

$$F'(x,y_1,y_2) = \begin{bmatrix} 1 & y_2^2 & 2y_1y_2 \\ 1 & 1 & 0 \end{bmatrix},$$

so

$$F'_x(x,y_1,y_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$F'_y(x,y_1,y_2) = \begin{bmatrix} y_2^2 & 2y_1y_2 \\ 1 & 0 \end{bmatrix},$$

thus

$$F'_y(-1,1,1) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix},$$

and the determinant equals 1, so the assumptions are met. Indeed, H exists. Moreover,

$$-(F'_y)^{-1} \cdot F'_x = -\begin{bmatrix} y_2^2 & y_1 \\ 1 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ 1/2y_1y_2 & -y_2/2y_1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 \\ \frac{-1+y_2^2}{2y_1y_2} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

(because since $H(x) = (-x,1)$ to $y_2 = 1$), which is exactly the same as

$$H' = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Meanwhile at $(0,0,1)$

$$F'_y(0,0,1) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

has zero determinant and manually we also will not be able to determine y from x, since y_1 may be equal to zero.

2. Consider function $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by formula $F(x,y) = (e^x \cos y, e^x \sin y)$. Check whether it is one-to-one. Does for any fixed (x,y) there exist $\delta > 0$ such that on $B((x,y),\delta)$ there exists G inverse to F? If so calculate G'.

   Obviously, it is not one-to-one $x = 0, y = 2k\pi,\ k \in \mathbb{Z}$ we get $(1,0)$.

   We have
$$F'(x,y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix},$$
   so
$$\det F'(x,y) = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} > 0,$$
   so by the theorem of inverse function there exists $\delta > 0$ such that $B((x,y),\delta)$ there exists G inverse to F.

   We also have
$$G' = (F')^{-1} = \begin{bmatrix} e^{-x} \cos y & e^{-x} \sin y \\ -e^{-x} \sin y & e^{-x} \cos y \end{bmatrix}.$$

3. Let $F(x,y,z) = x^2/4 + y^2/9 + z^2 - 1$. Check whether there exist r_1, r_2 and a function $h\colon B((2,0),r_1) \to \mathbb{R}$ such that $F(x,y,z) = 0$ for $(x,y) \in B((2,0),r_1)$ and $z \in (-r_2, r_2)$ if and only if $z = h(x,y)$. If it exists, find $h'(x,y)$.

   We have
$$F'(x,y,z) = [x/2, 2y/9, 2z],$$
   so $F'_z = 2z$, if $z \neq 0$, then $\det F'_z \neq 0$. For $x = 2$, $y = 0$ we have $1 + 0 + z^2 - 1 = 0$, so $z^2 = 0$. Thus such a function does not exist (we do not know the sign of z).

4. Let $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ and
$$F(x,y,z) = (x^2 + 2y^2 + 3z^2 - 6, x + y + z)$$
   Let $M = \{(x,y,z) \in \mathbb{R}^3 \colon F(x,y,z) = (0,0)\}$. Prove that in M at every point there exists a neighbourhood such that two of variables (x,y,z) can be determined as a function of the third one.

   We have
$$F'(x,y,z) = \begin{bmatrix} 2x & 4y & 6z \\ 1 & 1 & 1 \end{bmatrix},$$
   since we are asked about and two variables we calculate determinants of all submatrices 2×2:
$$\det \begin{bmatrix} 2x & 4y \\ 1 & 1 \end{bmatrix} = 2x - 4y$$
$$\det \begin{bmatrix} 2x & 6z \\ 1 & 1 \end{bmatrix} = 2x - 6z$$
$$\det \begin{bmatrix} 4y & 6z \\ 1 & 1 \end{bmatrix} = 4y - 6z$$
   By the implicit function theorem if the first determinant is non-zero then we can determine z, and if the second in non-zero, then we can determine y, and if the third is non-zero we can determine z. So it is not possible to determine any of these variables from the other two only if all the determinants are zero.

   But then $x = 2y$, and $6z = 4y$, so $z = 2y/3$. But if $F(x,y,z) = (0,0)$, then also $x+y+z = 2y+y+2y/3 = 0$, so $x = y = z = 0$, but then $x^2 + 2y^2 + 3z^2 - 6 = -6 \neq 0$ – a contradiction. So at every point of M one can determine at least one of the variables from the other two.

5. Consider equation $3x + e^x = y + e^y$.

   a) show that there exists a function $f(x)$, such that $f(0) = 0$ and $y = f(x)$ is a solution for $x \in I$, where I is an interval containing 0.
      We have $F(x,y) = 3x + e^x - y - e^y = 0$ and $a = (0,0)$. Then $F'(x,y) = [3 + e^x, -1 - e^y]$, so at $(0,0)$ we get $[3, -1]$, to determine y from x, we check that $-1 \neq 0$. Thus the hypothesis follows from the implicit function theorem.

b) show that it is possible to choose I in such a way that f is a diffeomorphism.
By the implicit function theorem

$$f'(x) = -(-1 - e^y)^{-1} \cdot (3 + e^x) = \frac{3 + e^x}{1 + e^y},$$

which at 0 equals $3 \neq 0$, so by the inverse function theorem there is an inverse function of C^1 class on some interval, so f jest is a diffeomorphism there.

c) find $(f^{-1})'$ at $y = 0$.
By inverse function theorem it is $1/f'$, i.e. $1/3$.

6. Show that the equations

$$F_1(x, y, t) = 3x^2y + t^2x - ty^2 - 2 = 0$$

and

$$F_2(x, y, t) = tx^2 + xy^2 - 2t^2y = 0$$

define x and y as functions of t, such that $x(1) = y(1) = 1$, assuming that t is close enough to 1. Find the direction of the tangent to the curve $\{(x(t), y(t), t) : t \in \mathbb{R}\}$ at $t = 1$.
We have $F(x, y, t) = (3x^2y + t^2x - ty^2 - 2, tx^2 + xy^2 - 2t^2y)$, so

$$F'(x, y, t) = \begin{bmatrix} 6xy + t^2 & 3x^2 - 2ty & 2tx - y^2 \\ 2tx + y^2 & 2xy - 2t^2 & x^2 - 4t \end{bmatrix},$$

which at $(1, 1, 1)$ is

$$F'(1, 1, 1) = \begin{bmatrix} 7 & 1 & 1 \\ 3 & 0 & -3 \end{bmatrix},$$

and since we want to determine x, y, we check that

$$\det \begin{bmatrix} 7 & 1 \\ 3 & 0 \end{bmatrix} = -3 \neq 0,$$

so the implicit function theorem can be applied. We get $h(t) = (x(t), y(t))$. We know that

$$h'(t) = -\begin{bmatrix} 6xy + t^2 & 3x^2 - 2ty \\ 2tx + y^2 & 2xy - 2t^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2tx - y^2 \\ x^2 - 4t \end{bmatrix},$$

which at $t = 1$ is

$$-\begin{bmatrix} 7 & 1 \\ 3 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 & -1 \\ -3 & 7 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 \\ -24 \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \end{bmatrix}.$$

So the direction of the tangent is $[x'(1), y'(1), 1]$, i.e. $(3, -8, 1)$.