Mathematical analysis 2, WNE, 2018/2019
meeting 19. – solutions

7 May 2019

1. Find the equation of the plane tangent to the surface described by the following equations at the indicated point.

   a) $\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} = 1$, $P = (1, -1, 1)$,

   $$f'(x, y, z) = \left(\frac{1}{3x^{2/3}}, \frac{1}{3y^{2/3}}, \frac{1}{3z^{2/3}}\right).$$

   Thus, $f'(1, -1, 1) = (1/3, 1/3, 1/3)$. So the equation for $T(M)$ is $x/3 + y/3 + z/3 = 0$, which is the same as $x + y + z = 0$, so the plane tangent to the surface is described by $x + y + z = 1$.

   b) $xyz + x^2 - 3y^2 + z^3 = 14$, $P = (5, -2, 3)$.

   $$f'(x, y, z) = \left(2x + yz, xz - 6y, xy + 3z^2\right).$$

   Thus, $f'(5, -2, 3) = (4, 27, 17)$. So the equation for $T(M)$ is $4x + 27y + 17z = 0$, so the plane tangent to the surface is described by $4x + 27y + 17z = 17$.

2. Find all points on the surface described $z = -x^2 - y^2 + 8x - 6y + 10$, at which the tangent plane is horizontal.

   $$f(x, y, z) = -x^2 - y^2 + 8x - 6y + 10 - z$$

   Horizontal planes are described by equations of form $z = a$, so we are looking for points at which two first coordinates of the gradient i.e. $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are zeroes.

   $$\frac{\partial f}{\partial x} = -2x + 8 = 0$$
   $$\frac{\partial f}{\partial y} = -2y - 6 = 0$$

   We get it at $x = 4, y = -3$, and then $z = -16 - 9 + 32 + 18 + 10 = 35$. So the only such point is $(4, -3, 35)$.

3. Determine all the points on the surface described by the equation

   $$z = \frac{3}{4}y^2 + \frac{1}{24}y^3 - \frac{1}{32}y^4 - x^2,$$

   where the tangent plane to the surface is horizontal. Which of these points are local extrema of the function $z(x, y)$?

   Similarly as before we are looking for points at which

   $$\frac{\partial f}{\partial x} - 2x = 0$$
   $$\frac{\partial f}{\partial y} = \frac{3}{2}y + \frac{1}{8}y^2 - \frac{1}{8}y^3 = 0$$

   So $x = 0$ and $y = 0$ or $12 + y - y^2 = 0$, thus $y = -3$ or $y = 4$. Then respectively $z = 0$, $z = 99/32$ and $z = 20/3$, thus these point are $(0, 0, 0)$, $(0, -3, 99/32)$ and $(0, 4, 20/3)$.

We now check whether the function $z(x,y)$ has extrema in $(0,0)$, $(0,-3)$, $(0,4)$. We know that the gradient is zero at these points. So we have to check second order partial derivatives. Their matrix is equal to

$$\begin{bmatrix} -2 & 0 \\ 0 & \frac{3}{2}+\frac{1}{4}y-\frac{3}{8}y^2 \end{bmatrix},$$

At $(0,0)$

$$\begin{bmatrix} -2 & 0 \\ 0 & \frac{3}{2} \end{bmatrix},$$

is non-definite, there is no local extremum at this point.

At $(0,-3)$

$$\begin{bmatrix} -2 & 0 \\ 0 & \frac{-21}{8} \end{bmatrix},$$

is negative definite, it is a local maximum.

and at $(0,4)$

$$\begin{bmatrix} -2 & 0 \\ 0 & \frac{-7}{2} \end{bmatrix},$$

is also negative definite, it is a local maximum.

4. Find a diffeomorphism between the following pairs of domains:

   a) the interior of the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$ the interior of the triangle with vertices $(0,0)$, $(0,1)$ and $(2,0)$,
   Clearly, $F(x,y)=(2x,y)$.

   b) the interior of the triangle with vertices $(0,0)$, $(0,1)$ i $(1,0)$ and the interior of the square with vertices $(0,0)$, $(0,1)$, $(1,1)$ i $(1,0)$,
   We stretch it accordingly: $F(x,y)=(x/y,y)$.

   c) the interior of the triangle with vertices $(0,0)$, $(0,1)$ i $(1,0)$ and the interior of the unit circle with centre $(0,0)$,
   First we transform it onto a square: $F(x,y)=(x/y,y)$, next we make it symmetrical with respect to $(0,0)$, so $G(x,y)=(2x-1,2y-1)$. Finally we squeeze it along radii. Maximal radii in a circle is 1, but in the square in direction (x,y) it is $m=\frac{\sqrt{x^2+y^2}}{\max(|x|,|y|)}$. So

$$H(x,y)=\begin{cases}(x/m,y/m) & ,\ \text{for } (x,y)\neq(0,0)\\ (0,0) & ,\ \text{for } (x,y)=(0,0)\end{cases}$$

   Thus,

$$D(x,y)=H(G(F(x,y))=H(G(x/y,y))=$$

$$=H(2x/y-1,2y-1)=\begin{cases}\frac{(2x/y-1,2y-1)\cdot\max(|2x/y-1|,|2y-1|)}{\sqrt{(2x/y-1)^2+(2y-1)^2}} & ,\ \text{for } (x,y)\neq(1/4,1/2)\\ (0,0) & ,\ \text{for } (x,y)=(1/2,1/4)\end{cases}$$

   d) the interior of the triangle with vertices $(0,0)$, $(0,1)$ i $(1,0)$ and the whole plane $\mathbb{R}^2$.
   First we map it into a square: $F(x,y)=(x/y,y)$, and next we apply function $f(x)=\frac{(x-1/2)}{x(1-x)}$, to both coordinates, which for $x\to0^+$ converges to $-\infty$ and for $x\to1^-$ converges to ∞, i.e. $G(x,y)=\left(\frac{(x-1/2)}{x(1-x)},\frac{(y-1/2)}{y(1-y)}\right)$, so we get the diffeomorphism

$$D(x,y)=G(F(x,y))=G(x/y,y)=\left(\frac{y(x/y-1/2)}{x(1-x/y)},\frac{(y-1/2)}{y(1-y)}\right).$$

5. Find a diffeomorphism $f\colon A\to B$, where

$$A=\{(x,y)\in\mathbb{R}^2\colon 1<x^2+y^2<4\},$$

$$B=\{(x,y)\in\mathbb{R}^2\colon 1<x^2+y^2<9\}.$$

We need a function which will map the interval of radii $(1, 2)$ onto $(1, 3)$. The function $f(r) = 2(r-1)+1 = 2r - 1$ does it. In other words we have to double the distance of every point from zero and next subtract the unit vector in its direction, so

$$F(x, y) = 2(x, y) - \frac{(x, y)}{\sqrt{x^2 + y^2}}.$$