Mathematical analysis 2, WNE, 2018/2019
meeting 18. – solutions

30 April 2019

1. Give an example of continuous functions of two variables, which has two local maxima, but no other local extrema.

   E.g.

$$f(x,y) = \begin{cases} -(x+1)^2 - y^2, & \text{dla } x < 0, \\ -(x-1)^2 - y^2, & \text{dla } x \geqslant 0. \end{cases}$$

2. Show that the function $2(1-e^{2y}+x^2)^3 - 3(1-e^{2y}+x^2)^2 - 24x^2e^{2y}$ has exactly one critical point at which the function has a strict local maximum, but the function is neither bounded above or below.

$$\frac{\partial f}{\partial x} = 12x(x^4 - (2x^2+5)e^{2y} + x^2 + e^{4y})$$

$$\frac{\partial f}{\partial y} = -12e^{2y}(-(2x^2+1)e^{2y} + (x^2+5)x^2 + e^{4y})$$

   Are zero, if $x = 0$ or $x^4 - (2x^2+5)e^{2y} + x^2 + e^{4y}$. In the first case $e^{2y} = e^{4y}$, so $y = 0$.

   In the second case as well $-(2x^2+1)e^{2y} + (x^2+5)x^2 + e^{4y} = 0$, We get

$$-4x^2e^{2y} - 6e^{2y} - 4x^2 = 0$$

   Thus, $4x^2(1+e^{2y}) = -6e^{2y}$, which is a contradiction.

   The only critical point is $(0,0)$.

   Second order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 12(-(6x^2+5)e^{2y} + (5x^2+3)x^2 + e^{4y})$$

$$\frac{\partial^2 f}{\partial x \partial y} = 24xw^{2y}(-2x^2 + 2e^{2y} - 5)$$

$$\frac{\partial^2 f}{\partial y^2} = -24e^{2y}(-2(2x^2+1)e^{2y} + (x^2+5)x^2 + 3e^{4y}),$$

   At $(0,0)$ are respectively $12(-5+1) = -48$, 0 and $-24(-2+3) = -24$, and we get matrix

$$\begin{bmatrix} -48 & 0 \\ 0 & -24 \end{bmatrix}$$

   which is negative definite, so it is a maximum.

   For $x = 0$, we get $f(0,y) = 2(1-e^{2y})^3 - 3(1-e^{2y})^2 = 2t^3 - 3t^2$ for $t = 1 - e^{2y} \in (-\infty, 1)$ is unbounded from below. But for $y = 0$, we get $f(x,0) = 2x^6 - 3x^4 - 24x^2$ which is unbounded from above.

3. Show that there is no function $f(x,y)$ of C^2 class such that $\frac{\partial f}{\partial x}(x,y) = 6xy^2$ and $\frac{\partial f}{\partial x}(x,y) = 8x^2y$.

   Then $\frac{\partial^2 f}{\partial x \partial y}(x,y) = 12xy$, but $\frac{\partial^2 f}{\partial x \partial y}(x,y) = 8x^2$, are different which is not possible.

4. Determine if the following functions satisfies Laplace's equation:

$$\frac{\partial f^2}{\partial x^2} + \frac{\partial f^2}{\partial y^2} = 0.$$

a) $f(x,y) = \sqrt{x^2+y^2}$,

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} \\
\frac{\partial f}{\partial x} &= \frac{y}{\sqrt{x^2+y^2}} \\
\frac{\partial^2 f}{\partial x^2} &= \frac{y^2}{(x^2+y^2)^{3/2}} \\
\frac{\partial^2 f}{\partial y^2} &= \frac{x^2}{(x^2+y^2)^{3/2}}
\end{aligned}$$

But

$$\frac{y^2}{(x^2+y^2)^{3/2}} + \frac{x^2}{(x^2+y^2)^{3/2}} = \frac{1}{\sqrt{x^2+y^2}} \neq 1.$$

b) $f(x,y) = \ln(\sqrt{x^2+y^2})$,

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{x}{x^2+y^2} \\
\frac{\partial f}{\partial x} &= \frac{y}{x^2+y^2} \\
\frac{\partial^2 f}{\partial x^2} &= \frac{y^2-x^2}{(x^2+y^2)^2} \\
\frac{\partial^2 f}{\partial y^2} &= \frac{x^2-y^2}{(x^2+y^2)^2}
\end{aligned}$$

And indeed

$$\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0 \neq 0.$$

c) $f(x,y) = e^{-x}\sin y$.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -e^{-x}\sin y \\
\frac{\partial f}{\partial x} &= e^{-x}\cos y \\
\frac{\partial^2 f}{\partial x^2} &= e^{-x}\sin y \\
\frac{\partial^2 f}{\partial y^2} &= -e^{-x}\sin y
\end{aligned}$$

And indeed

$$e^{-x}\sin y - e^{-x}\sin y = 0.$$

5. Find and classify all the critical points of the following functions:

a) $f(x,y) = e^{xy} - 2xy$,
We get

$$\begin{aligned}
\frac{\partial f}{\partial x} &= y(e^{xy}-2), \\
\frac{\partial f}{\partial y} &= x(e^{xy}-2).
\end{aligned}$$

For $x = 0$ we also have $y = 0$. If $x \neq 0$, to $xy = \ln 2$. In the second case the value is constant, so it is not an extremum. Let us consider $(0,0)$. Here also there is not extremum, because for $x = 0$ the function is constant.

b) $f(x, y, z) = x^2 + y^2 + z^2 - xy + x + 2z$.
We get

$$\frac{\partial f}{\partial x} = 2x - y + 1,$$
$$\frac{\partial f}{\partial y} = 2y - x,$$
$$\frac{\partial f}{\partial z} = 2z + 2.$$

Are 0 if $z = -1$, $x = 2y$, so $y = -1/3$, $x = -2/3$.

$$\frac{\partial^2 f}{\partial x^2} = 2,$$
$$\frac{\partial^2 f}{\partial y^2} = 2,$$
$$\frac{\partial^2 f}{\partial z^2} = 2,$$
$$\frac{\partial^2 f}{\partial x \partial y} = -1,$$
$$\frac{\partial^2 f}{\partial y \partial z} = 0.$$
$$\frac{\partial^2 f}{\partial z \partial x} = 0.$$

The second order derivative matrix:
$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

By Sylvester's criterion (the determinants are $2, 1$ and 2) is positive definite, so at $(-1, -1/3, -2/3)$ there is a local minimum.