Mathematical analysis 2, WNE, 2018/2019
meeting 17. – solutions

25 April 2019

1. Find and classify the local extrema of the following functions:

   a) $f(x, y) = 3x^2 + 6xy + 2y^3 + 12x - 24y$,

$$\frac{\partial f}{\partial x} = 6x + 6y + 12,$$
$$\frac{\partial f}{\partial y} = 6x + 6y^2 - 24,$$

   and are zero, if $x = -2 - y$, thus $6y^2 - 6y - 36 = 0$, so $(y - 3)(y + 2) = 0$, and therefore the critical points are $(-5, 3)$ and $(0, -2)$.
   Second order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 6,$$
$$\frac{\partial^2 f}{\partial x \partial y} = 6,$$
$$\frac{\partial^2 f}{\partial y^2} = 12y.$$

   Thus, Hessian at $(0, -2)$ takes form

$$\begin{bmatrix} 6 & 6 \\ 6 & -24 \end{bmatrix}$$

   and this form is non-definite, because $d^2(h_x, h_y) = 6h_x^2 + (6 + 6)h_x h_y - 12h_y^2$ takes value > 0 e.g. for $h_x = 1, h_y = 0$ and value < 0 for $h_x = 0, h_y = 1$. Thus, it is not an extremum.
   Hessian at $(-5, 3)$ takes form

$$\begin{bmatrix} 6 & 6 \\ 6 & 36 \end{bmatrix}$$

   by Sylvester's criterion ($\det A_1 = 6$, $\det A_2 = 180$) is positive definite, so it is a minimum.

   b) $f(x, y) = x^3 y - 3xy^2$.

$$\frac{\partial f}{\partial x} = 3x^2 y - 3y^2,$$
$$\frac{\partial f}{\partial y} = x^3 - 6xy,$$

   Both are equal to zero, if $x = 0$ and then $y = 0$. If $x \neq 0$, then $x^2 = 6y$, thus $18y^2 - 3y^2 = 0$, therefore $15y^2 = 0$ i $y = 0$, but then $x = 0$, a contradiction. Therefore, the only critical point is $(0, 0)$. But it is not a local extremum since the function takes value 0 for any (x, y) such that $x = 0$.

2. Show that the function $f(x, y) = x^2(1 + y)^3 + y^2$ has exactly one critical point p. Show that p is a local minimum and $\sup_{(x,y)\in\mathbb{R}^2} f(x, y) = +\infty$ and $\inf_{(x,y)\in\mathbb{R}^2} f(x, y) = -\infty$.

$$\frac{\partial f}{\partial x} = 2x(1 + y)^3,$$

$$\frac{\partial f}{\partial y} = 3x^2(1+y)^2 + 2y,$$

Both are equal zero, if $x = 0$ and then $y = 0$, but if $x \neq 0$, then $(1+y)^3 = 0$, and $y = -1$, but then $\frac{\partial f}{\partial y}$ is not equal to zero. So the only critical point is $(0,0)$.

Second order derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 2(1+y)^3,$$
$$\frac{\partial^2 f}{\partial x \partial y} = 6x(1+y)^2,$$
$$\frac{\partial^2 f}{\partial y^2} = 6x^2(1+y) + 2.$$

Hessian at $(0,0)$ is

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

so by Sylvester's criterion ($\det A_1 = 2$, $\det A_2 = 4$) it is positive definite, so it is a minimum.

We have

$$\lim_{y\to-\infty} f(1,y) = \lim_{y\to-\infty} (1+y)^3 + y^2 = -\infty.$$

and

$$\lim_{y\to\infty} f(1,y) = \lim_{y\to\infty} (1+y)^3 + y^2 = \infty.$$

Thus, $\sup_{(x,y)\in\mathbb{R}^2} f(x,y) = +\infty$ and $\inf_{(x,y)\in\mathbb{R}^2} f(x,y) = -\infty$.

3. The P, V and T denote pressure P, volume V, and temperature T of a given gas, which satisfies the equation $PV = RT$, where R is a certain constant. Prove that

$$\frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} = -1.$$

Thus,

$$P(V,T) = \frac{RT}{V},$$
$$\frac{\partial P}{\partial V} = -\frac{RT}{V^2},$$
$$V(P,T) = \frac{RT}{P},$$
$$\frac{\partial V}{\partial T} = \frac{R}{P},$$
$$T(V,P) = \frac{PV}{R},$$
$$\frac{\partial T}{\partial P} = \frac{V}{R},$$

so

$$\frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} = -\frac{RTRV}{V^2PR} = -\frac{RT}{PV} = -1.$$

4. Find the maximal and minimum value of

$$f(x,y) = \sin x + \sin y + \sin(x+y)$$

on the set $D = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x, y \leqslant \pi/2\}$.

We check the critical points

$$\frac{\partial f}{\partial x} = \cos x + \cos(x+y),$$
$$\frac{\partial f}{\partial y} = \cos y + \cos(x+y),$$

Thus, $\cos x = -\cos(x+y) = \cos y$, but since $0 \leqslant x, y \leqslant \pi/2$, then $x = y$, and $x+y = 2x = \pi/2+(\pi/2-x) = \pi - x$, thus $3x = \pi$ and $x = y = \pi/3$. We have $f(\pi/3, pi/3) = 3\sqrt{3}/2$.

For $x = 0$ we get $f(y) = 2\sin y$, and so $2\cos y = 0$, so there is no critical point in the interval $(0, \pi/2)$. Analogously, for $y = 0$ we get $f(x) = 2\sin x$, thus $2\cos x = 0$, so there is no critical point in the interval $(0, \pi/2)$. For $x = \pi/2$ we have $f(y) = 1+\sin y+\sin(\pi/2+y)$, thus in a critical point $\cos y = -\cos(\pi/2+y)$, $y = \pi/4$ and the value is: $f(\pi/2, pi/4) = 1+\sqrt{2}$. Similarly, for $y = \pi/2$ we have $f(x) = 1+\sin x+\sin(\pi/2+x)$, thus in a critical point $\cos x = -\cos(\pi/2+x)$, $x = \pi/4$ and the value is: $f(\pi/4, pi/2) = 1+\sqrt{2}$.

In the vertices $f(0,0) = 0$, $f(0, \pi/2) = 2$, $f(\pi/2, 0) = 2$, $f(\pi/2, \pi/2) = 2$. Since $2 < 1+\sqrt{2} < 3\sqrt{3}/2$, the maximum value is $3\sqrt{3}/2$ at $(\pi/3, pi/3)$, and minimum 0 is taken at $(0,0)$.

5. Among triangles inscribed inside a circle of radius R find the one with maximal area.

   Without a loss of generality we can assume that the base line is horizontal and then to maximise the height of the triangle we take the third vertex at $(0, R)$. The only parameter is a, such $y = a$ is the base line, $a \in [-R, R]$. Then $h = R - a$, and the length of the base is $2\sqrt{R^2 - a^2}$, thus the surface area is $P(a) = (R-a)\sqrt{R^2-a^2}$, and

$$P'(a) = -\sqrt{R^2-a^2} - \frac{a(R-a)}{\sqrt{R^2-a^2}} = \frac{2a^2 - aR - R^2}{\sqrt{R^2-a^2}} = \frac{(a-R)(2a+R)}{\sqrt{R^2-a^2}},$$

   which is equal to zero if $a = R$ and $a = -R/2$. The second value corresponds to the maximal area, thus

$$P(-R/2) = 3R/4 \cdot \sqrt{3}R/2 = \frac{3R^2\sqrt{3}}{8}.$$

6. Among rectangular boxes inscribed inside a sphere of radius R find the one with maximal volume.

   Let the upper side be $y = a$, and the right side $x = b$, $a, b \in (0, R]$. Then the lengths of the edges are $2a, 2b$ and $2\sqrt{R^2-a^2-b^2}$, thus

$$V(a,b) = 8ab\sqrt{R^2-a^2-b^2}.$$

   Partial derivatives:

$$\frac{\partial V}{\partial a} = 8b\sqrt{R^2-a^2-b^2} - \frac{8a^2b}{\sqrt{R^2-a^2-b^2}} = \frac{8bR^2 - 16ba^2 - 8b^3}{\sqrt{R^2-a^2-b^2}},$$

$$\frac{\partial V}{\partial b} = 8a\sqrt{R^2-a^2-b^2} - \frac{8ab^2}{\sqrt{R^2-a^2-b^2}} = \frac{8aR^2 - 16ab^2 - 8a^3}{\sqrt{R^2-a^2-b^2}},$$

   Since $a = 0$ or $b = 0$ are not in the domain we get $R - 2a^2 - b^2 = 0$ and $R - 2b^2 - a^2 = 0$, so $R - 2(R - 2a^2) - a^2 = 0$, thus $R^2/3 = a^2$, so $a = R\sqrt{3}/3 = b$, and then the volume is

$$8\frac{R^2}{3}\sqrt{R^2/3} = \frac{8R^3\sqrt{3}}{9}.$$