Mathematical analysis 2, WNE, 2018/2019
meeting 16. – solutions

16 April 2019

1. Let
$$f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{, for } (x,y)\neq(0,0),\\ 0 & \text{, for } (x,y)=(0,0).\end{cases}$$
Show that

   a) the point $(0,0)$ is a critical point of the function,
   $$\lim_{h\to 0}\frac{\frac{0}{h^2}-0}{h}=0,$$
   so $\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial y}(0,0)=0$, thus indeed it is a critical point.

   b) all second order partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x\partial y}$ and $\frac{\partial^2 f}{\partial y\partial x}$ exist at $(0,0)$, but
   $$\frac{\partial^2 f}{\partial x\partial y}(0,0)\neq\frac{\partial^2 f}{\partial y\partial x}(0,0).$$
   We have to calculate first order derivatives in all the other points
   $$\frac{\partial f}{\partial x}(x,y)=\begin{cases}\dfrac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} & \text{, for } (x,y)\neq(0,0),\\ 0 & \text{, for } (x,y)=0.\end{cases}$$
   $$\frac{\partial f}{\partial y}(x,y)=\begin{cases}\dfrac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2} & \text{, for } (x,y)\neq(0,0),\\ 0 & \text{, for } (x,y)=0.\end{cases}$$
   So the second order derivatives at $(0,0)$ are
   $$\frac{\partial^2 f}{\partial x^2}(0,0)=\lim_{h\to 0}\frac{\frac{0}{h^4}-0}{h}=0,$$
   $$\frac{\partial^2 f}{\partial x\partial y}(0,0)=\lim_{h\to 0}\frac{\frac{-h^5}{h^4}-0}{h}=-1,$$
   $$\frac{\partial^2 f}{\partial y\partial x}(0,0)=\lim_{h\to 0}\frac{\frac{h^5}{h^4}-0}{h}=1,$$
   $$\frac{\partial^2 f}{\partial y^2}(0,0)=\lim_{h\to 0}\frac{\frac{0}{h^4}-0}{h}=0.$$

   c) the point $(0,0)$ is not a local extremum of f.
   Definitely, the function is constant for $x=0$.

2. Let $f(x,y)=(y-x^2)(y-3x^2)$. Show that

a) $f'(0,0) = (0,0)$,

$$\frac{\partial f}{\partial x} = -2x(y - 3x^2) - 6x(y - x^2),$$
$$\frac{\partial f}{\partial y} = (y - 3x^2) + (y - x^2),$$

which for $x = y = 0$ is $(0,0)$.

b) for every $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$, the function $h(t) = f(ta, tb)$ has a local minimum for $t = 0$, $h(t) = f(ta, tb) = (tb - t^2a^2)(tb - 3t^2a^2)$, thus $h'(t) = 12a^4t^3 - 12a^2bt^2 + 2b^2t$, which for $t = 0$ is 0, so it is a critical point $h''(t) = 36a^4t^2 - 24a^2bt + 2b^2$ for $t = 0$ is $2b^2$. For $b \neq 0$ it is > 0, and thus we have a minimum. For $b = 0$: $h''(t) = 36a^4t^2$, so $h'''(t) = 72a^4t$ equals zero for $t = 0$, a $h^{(4)}(t) = 72a^2 > 0$ (since $b = 0$, $a \neq 0$), so we also have a minimum.

c) f does not have a local minimum at $(0,0)$.
For $y = x^2$ the function is constant and equal to zero.

3. Let $A = \{(x,y,z) \in \mathbb{R}^2 \colon 2x - 3y + z = 1\}$. Find a point $p \in A$ closest to $(3,-2,1)$.
$z = (1 - 2x + 3y)$, so the square of the distance between (x,y,z) and p is

$$d(x,y) = (x-3)^2 + (y+2)^2 + (-2x+3y)^2$$

and

$$\frac{\partial d}{\partial x} = 2(5x + 6y - 3),$$
$$\frac{\partial d}{\partial x} = 4(-3x + 5y + 1),$$

Both are equal zero for $(x,y) = (-9/7, -11/7)$, so the minimum is $(-9/7, -11/7, -8/7)$.

4. Find the maximum possible volume of a cylinder whose height plus circumference of the base does not exceed 108cm.

$2r + h = 108$, so $h = 108 - 2r$, thus $V(r) = 2\pi r(108 - 2r) = -4\pi r^2 + 216\pi r$, and $V'(r) = -8\pi r + 216\pi$, therefore for $r = 27$ $V' = 0$. It is when the volume is greatest (negative second order derivative) and is equal to 2916π cm^3.

5. Find and classify the local extrema of the following functions:

a) $f(x,y) = x^3 + y^3 + 3xy + 3$,
Partial derivatives are $3x^2 + 3y$ and $3y^2 + 3x$, are equal to zero if $y = -x^2$, so $3x^4 + 3x = 0$, i.e. for $x = y = 0$ or $x = -1, y = -1$. The matrix of second order derivative is

$$\begin{bmatrix} 6x & 3 \\ 3 & 6y \end{bmatrix},$$

at $(0,0)$ it is

$$\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix},$$

and is non-definite so there is no extremum in this critical point. For $(-1,-1)$ we get

$$\begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix},$$

And the form is negative definite, so we have a local maximum here.

b) $f(x,y) = e^{-x^4-y^4}$.
Partial derivatives are $-3x^3e^{-x^4-y^4}$ and $-3y^3e^{-x^4-y^4}$, are equal to zero if $y = x = 0$. The matrix of second order derivative is

$$\begin{bmatrix} -9x^2e^{-x^4-y^4} + 9x^6e^{-x^4-y^4} & 9x^3y^3e^{-x^4-y^4} \\ 9x^3y^3e^{-x^4-y^4} & -9y^2e^{-x^4-y^4} + 9y^6e^{-x^4-y^4} \end{bmatrix},$$

at $(0, 0)$ it is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

it does not tell us whether there is a local extremum there, but we see that the power of e becomes more negative if we go away from $(0, 0)$, so it is a maximum – the only point at which the function reaches value 1.