Mathematical analysis 2, WNE, 2018/2019
meeting 15. – solutions

11 April 2019

1. Find the maximal value of function:

   a) $z(x,y) = 1 + \frac{4}{3}x^3 + 4y^3 - x^4 - y^4$,
   Partial derivatives: $4x^2 - 4x^3$ and $12y^2 - 4y^3$. Are zero for $x = 0$ or $x = 1$ oraz $y = 0$ or $y = 3$. We check the values in those four critical points: $z(0,0) = 1$, $z = (0,3) = 28$, $z(1,0) = 4/3$ i $z(1,3) = 28\frac{1}{3}$. Since the function converges to $-\infty$, if $\|(x,y)\| \to \infty$ and is continuous, the maximal value equals $28\frac{1}{3}$.

   b) $z(x,y) = (1+x^2)\exp(-x^2-y^2)$.
   Partial derivatives are: $2xe^{-x^2-y^2} - 2x(1+x^2)e^{-x^2-y^2} = -2x^3e^{-x^2-y^2}$ and $-2y(1+x^2)e^{-x^2-y^2}$, and are zero for $x = y = 0$. Then, $z(0,0) = 1$. Notice also that the function converges to 0, for $\|(x,y)\| \to \infty$ and is continuous, so the maximal value is 1.

2. Find dimensions x, y, z of a rectangular box with volume $V = 1000$ and minimal possible surface are. Does it make sense to ask about the maximal possible surface area?

   $z = 1000/xy$, the surface area is $S(x,y) = 2xy + 2yz + 2xz = 2xy + 2000/x + 2000/y$ and partial derivatives are $2y - 2000/x^2$ and $2x - 2000/y^2$, which are zero, for $y = 1000/x^2$ and $x = 1000/y^2$, so $x = 1000 \cdot x^4/1000000$, thus for $x = x^4/1000$, i.e. for $x = 10$ ($x = 0$ is not in the domain) and $y = 10$. Therefore, the minimal surface area we obtain for $x = y = z = 10$. There is no maximal surface area, the function grows to infinity.

3. Find dimensions x, y, z of a rectangular box of maximal possible volume and surface area of $600cm^2$.

   $2xy + 2yz + 2xz = 600$, so $z(x+y) = 300 - 2xy$, thus $z = \frac{300-xy}{x+y}$. Therefore,

   $$V(x,y) = \frac{300xy - x^2y^2}{x+y}.$$

   Partial derivatives are

   $$\frac{-2y^2(x^2+2xy-300)}{(x+y)^2}$$

   and

   $$\frac{-2x^2(y^2+2xy-300)}{(x+y)^2},$$

   thus $x^2 + 2xy = 300$ and $y^2 + 2xy = 300$, thus $x^2 - y^2 = 0$, but since $x, y > 0$ we get $x = y$, so $3x^2 = 300$, i.e. $x = y = z = 10$ which implies that the maximal volume is 1000, since if $\|(x,y)\| \to 0$ or $\to \infty$, then $V \to 0$.

4. A rectangular box without a lid has volume of 4 litres. What dimensions x, y, z to minimize the surface area of the sides?

   $z = 4000/xy$, so the surface area is $S(x,y) = xy + 2xz + 2yz = xy + 8000/x + 8000/y$. Partial derivatives: $y - 8000/x^2$ and $x - 8000/y^2$ are zero for $y = 8000/x^2$, so $x = x^4/8000$, thus $x^3 = 8000$, and $x = 20$. So $y = 20$ and $z = 10$.

5. Rectangular box is to have volume of 48 litres. The cost of material is $1PLN$ per m^2 of a side wall, $2PLN$ per m^2 of a lid and $3PLN$ per m^2 of the base. Determine the minimal cost of such a box.

   Thus $xyz = 48$ w dm. $z = 48/xy$, so the cost (in $0,01PLN$) is

   $$K(x,y) = 2xz + 2yz + 2xy + 3xy = 96/y + 96/x + 5xy.$$

Partial derivatives $-96/x^2+5y$ and $-96/y^2+5x$, are equal zero for $y=96/5x^2$, $5x=96\cdot 25x^4/96^2$, thus ($x=0$ is not in the domain) for $x^3=96/5$, i.e. $x=\sqrt[3]{96/5}$ and $y=\sqrt[3]{96^3/5^3\cdot 5^2/96^2}=\sqrt[3]{96/5}$ and then

$$K(\sqrt[3]{96/5},\sqrt[3]{96/5})=3\sqrt[3]{96^2\cdot 5}=\sqrt[3]{46080}/100PLN.$$