Mathematical analysis 2, WNE, 2018/2019
meeting 14. – solutions

9 April 2019

1. Calculate the partial derivatives of first and second order of

$$f(x,y) = x^2 - 3xy^2 + 2y^3 + 2y.$$

$$\frac{\partial f}{\partial x}(x,y) = 2x - 3y^2,$$
$$\frac{\partial f}{\partial y}(x,y) = -6xy + 6y^2 + 2,$$
$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2,$$
$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = -6y,$$
$$\frac{\partial^2 f}{\partial y^2}(x,y) = -6x + 12y.$$

2. Check whether point $(0,0)$ is a local extremum of:

   a) $z(x,y) = x^2 + y^2$,
   Obviously, yes, it is a local minimum and the only point in which the value is 0.

   b) $z(x,y) = x^2 - y^2$.
   Obviously, not. The function decreases along $x = 0$, and increases along $y = 0$.

3. Determine the extrema of the function

$$f(x,y,z) = x^2 - 2x - y^3 + 3y + 5z^2.$$

Partial derivatives: $\frac{\partial f}{\partial x} = 2x - 2$, $\frac{\partial f}{\partial y} = -3y^2 + 3$, $\frac{\partial f}{\partial z} = 10z$. They are zero for $x = 1, y = \pm 1, z = 0$, and these are candidates for extrema: $(1,1,0)$ i $(1,-1,0)$. We calculate second order derivatives to verify them. $\frac{\partial f}{\partial x^2} = 2$, $\frac{\partial f}{\partial x \partial y} = 0$, $\frac{\partial f}{\partial x \partial z} = 0$, $\frac{\partial f}{\partial y^2} = -6y$, $\frac{\partial f}{\partial y \partial z} = 0$, $\frac{\partial f}{\partial z^2} = 10$.

Thus $d^2 f = 2h_1^2 - 6yh_2^2 + 10h_3^2$, which at $(1,1,0)$ gives $2h_1^2 - 6h_2^2 + 10h_3^2$, which is positive for $h_1 = 1, h_2 = h_3 = 0$ and negative for $h_1 = h_3 = 0$ and $h_2 = 1$, so it is not an extremum.

However, at $(1,-1,0)$, it gives $2h_1^2 + 6h_2^2 + 10h_3^2$, which is always positive so here we get a minimum $f(1,-1,0) = -3$.

4. Does $f(x,y,z) = xy + yz + zx$ have local extrema?

Partial derivatives are $y+z, x+z, y+x$. If all are equal to zero, we get $x = -y$, $y = z$, so $x = y = z = 0$. And it is not an extremum since the function is constant for $x = y = 0$.

5. Find $\sup_{(x,y)\in D} f(x,y)$ and $\inf_{(x,y)\in D} f(x,y)$ for

   a) $f(x,y) = \sqrt{x^2+y^2}$, $D = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 \leqslant 1\}$,
   Obviously, $\inf_{(x,y)\in D} f(x,y) = 0$ which is a value for $(x,y) = (0,0)$, and $\sup_{(x,y)\in D} f(x,y) = 1$ which is the value at the boundary of D.

b) $f(x,y) = xy^2$, $D = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 \leqslant 3\}$,
We check whether there is an extremum. Partial derivatives are y^2 and $2xy$, and are both 0 for $y = 0$. The function takes then value 0.
Meanwhile for $y^2 = 3 - x^2$, $y = \pm\sqrt{3 - x^2}$ and $x \in [-\sqrt{3}, \sqrt{3}]$ we get $f(x) = x(3 - x^2) = -x^3 + 3x$, and has derivative equal to $-3x^2 + 3$, and is 0 for $x = 1$. Then $f(x) = -1 + 3 = 2$. For points $x = \pm\sqrt{3}$, the value is 0. Thus $\sup_{(x,y)\in D} f(x,y) = 2$ and $\inf_{(x,y)\in D} f(x,y) = 0$

c) $f(x,y) = x^2 + y^2 - x - y$, D is a triangle with vertices $(0,0)$, $(0,2)$ and $(2,0)$,
We check the partial derivatives: $2x - 1, 2y - 1$. They are zero for $x = y = 1/2$. It is a point in D and the value there is

$$1/4 + 1/4 - 1/2 - 1/2 = -1/2.$$

But we also need to check on the edges of the triangle. One edge is given by $x = 0$ and then $f(y) = y^2 - y$ has derivative $2y - 1$, which is 0 for $y = 1/2$ and value $-1/4$. Similarly for the edge $y = 0$ we get $f(x) = x^2 - x$ with derivative $2x - 1$ which is 0 for $x = 1/2$ and then it takes value $-1/4$. The third edge is $y = x$ and then we get $f(x) = 2x^2 - 2x$, the derivative is $4x - 2$, so $x = 1/2$ and it is the point considered before.
Now the vertices $f(0,0) = 0, f(0,2) = 2,\ f(2,0) = 2$.
Thus, $\sup_{(x,y)\in D} f(x,y) = 2$ and $\inf_{(x,y)\in D} f(x,y) = -1/2$.

d) $f(x,y) = x^2 + y^2 - x$, D is a square with vertices $(\pm 1, \pm 1)$.
The partial derivatives are $2x - 1$ and $2y$, which equals zero for $(1/2, 0)$. At this point the value is $1/4 - 1/2 = -1/2$. Edges:

- $x = -1$, to $f(y) = y^2 + 2$, extremum for $y = 0$ equal to 2,
- $x = 1$, to $f(y) = y^2$, extremum for $y = 0$ equal to 0,
- $y = -1$, to $f(y) = x^2 - x + 1$, extremum for $x = 1/2$ equal to $3/4$,
- $x = 1$, to $f(y) = x^2 - x + 1$, extremum for $x = 1/2$ equal to $3/4$,

and the values at vertices $(1,1), (1,-1), (-1,1)$ and $(1,1)$ are respectively $1, 1, 3, 3$.
Thus $\sup_{(x,y)\in D} f(x,y) = 3$ and $\inf_{(x,y)\in D} f(x,y) = -1/2$.