Mathematical Analysis II, Department of Economics
academic year 2018/19, summer semester, B – solutions
Colloquium I, April 4, 2019

1. Calculate the indefinite integral $\displaystyle\int \frac{1}{(x-1)x^2}\,dx$. **answer:**..

$$\int \frac{1}{(x-1)x^2}\,dx = \int \left(\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2}\right)dx = \ln|x-1| - \ln|x| + \frac{1}{x} + C.$$

2. Calculate the definite integral $\displaystyle\int_1^e x^{-1}\cos(\ln(x))\,dx$. **answer:**..

$$\int_1^e x^{-1}\cos(\ln(x))\,dx = \in_0^1 \cos t\,dt = \sin t|_0^1 = \sin 1$$

3. Calculate the improper integral $\displaystyle\int_0^\infty x^2 e^{-x}\,dx$. **answer:**..

By parts you can get that:

$$\int x^2 e^{-x}\,dx = -e^{-x}(x^2+2x+2) + C,$$

so

$$\int_0^\infty x^2 e^{-x}\,dx = \lim_{b\to\infty}(-e^{-x}(x^2+2x+2))|_0^b = \lim_{b\to\infty} \frac{-(b^2+2b+2)}{e^b} + 2 = 0 + 2 = 2.$$

4. Is the set $\{(x,y)\in\mathbb{R}^2 : x^2+y^2 \leqslant 1,\ x \leqslant y\}$:

open	**Yes/No**: ....
closed	**Yes/No**: ....
bounded	**Yes/No**: ....
compact	**Yes/No**: ....
connected	**Yes/No**: ....
convex	**Yes/No**: ....

It is the upper left half of the unit circle with the boundary of the circle and with the line. Thus it is bounded, connected, convex, closed and compact but not open.

5. Find the following limit or state that it does not exist:

$$\lim_{n\to\infty}\left(\left(1+\frac{3}{n}\right)^n,\ \frac{\ln n}{3n},\ \sqrt[n]{3n}\right),$$

answer: ....

The limit is $(e^3, 0, 1)$.

6. Find the following limit or state that it does not exist:

$$\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^4+y^2}$$

answer: ....

$$\frac{x^2y^2}{x^4+y^2}=\frac{1}{x^2/y^2+1/x^2}\leqslant\frac{1}{1/x^2}=x^2\to 0.$$

7. Determine whether the function $f:\mathbb{R}^2\to\mathbb{R}$,

$$f(x,y)=\begin{cases}(x^2+y^2)\arctan\left(\dfrac{1}{x^2+y^2}\right) & \text{for } (x,y)\neq(0,0)\\ 0 & \text{for } (x,y)=(0,0)\end{cases}$$

is continuous. **Yes/No:**...................

For $r=\sqrt{x^2+y^2}$ we get

$$\lim_{(x,y)\to(0,0)}(x^2+y^2)\arctan\left(\frac{1}{x^2+y^2}\right)=\lim_{r\to 0}\frac{\arctan 1/r^2}{1/r^2}=\lim_{r\to 0}\frac{-2\frac{1}{r^3}\cdot\frac{1}{1/r^4+1}}{-2\frac{1}{r^3}}=0,$$

so it is continuous.

8. Let $f:\mathbb{R}^3\to\mathbb{R}$ be the function given by the formula

$$f(x,y,z)=e^{2x+2xy+z^2-z}.$$

Which unit vector v gives the direction of maximum rate of increase of f at the point $(0,0,0)$?

answer: ....

$$f'(x,y,z)=[(2+2y)e^{x+2xy+z^2},2xe^{x+2xy+z^2},(2z-1)e^{x+2xy+z^2}],$$
$$f'(0,0,0)=[2,0,-1],$$
$$v=\frac{[2,0,-1]}{\sqrt{5}}.$$

9. Let f : be the function described in question 8. Give the value of the directional derivative

$$\frac{\partial f}{\partial v}(0,0,0)$$

where $v = (1,2,3)$.

answer: ....

$$\langle (2,0,-1),(1,2,3)\rangle = -1.$$

10. Let (u,v) denote coordinates on $\mathbb{R}^2$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function described in the coordinates (u,v). If $g(x,y) = f(x+y, x-y)$, then find A and B such that

$$\frac{\partial g}{\partial x}(x,y) + \frac{\partial g}{\partial y}(x,y) = A\frac{\partial f}{\partial u}(u,v) + B\frac{\partial f}{\partial v}(u,v).$$

Hint: Compute $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ using the chain rule.

answer: ....

$$h(x,y) = (x+y, x-y),$$

so

$$h' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

so

$$g' = \left[\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}, \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v}\right].$$

Thus,

$$\frac{\partial g}{\partial x}(x,y) + \frac{\partial g}{\partial y}(x,y) = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} + \left(\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v}\right) = 2\frac{\partial f}{\partial u}.$$

Thus, $A = 2, B = 0$.