Mathematical analysis 2, WNE, 2018/2019
meeting 9. – solutions

19 March 2019

1. Check whether the limit exists. If it does find it.

   a) $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{\sqrt{x^2+y^2}}$,
   We can estimate
   $$\left|\frac{x^2-y^2}{\sqrt{x^2+y^2}}\right| = \frac{|x^2-y^2|}{\sqrt{x^2+y^2}} \leqslant \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} = r \to 0,$$
   thus $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = 0$.

   b) $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$,
   For $x_n = y_n = 1/n$ we get a constant sequence equal to 0, but for $x_n = 1/n$, $y_n = 0$ we get a constant sequence equal to 1. Thus the limit does not exist.

   c) $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy+yz+zx}{x^2+y^2+z^2}$.
   For $x_n = y_n = z_n = 1/n$ we get a constant sequence equal to 1, but for $x_n = y_n = 1/n$, $z_n = -1/n$ we get a constant sequence equal to $-1/3$. Thus the limit does not exist.

2. Determine whether the functions are continuous.

   a) $f(x,y) = \begin{cases} \dfrac{x-y}{x^3-y} & \text{, for } y \neq x^3 \\ 1 & \text{, for } y = x^3, \end{cases}$
   Obviously, the function is continuous at (x,y) such that $y \neq x^3$. Let (x_0,y_0) be such that $y_0 = x_0^3$. Let $x_n = x_0 + 1/n$, $y_n = x_0^3 + 1/n$. Then
   $$\frac{x-y}{x^3-y} = \frac{x_0^3 - x_0}{3x_0^2/n + 3x_0/n^2 + 1/n^3 - 1/n},$$
   which converges to $+\infty$, if $x_0^3 - x_0 \neq 0$, i.e. if $x_0 \neq -1, 0, 1$. Thus, the function is not continuous in these points. If $x_0 = -1, 0$ or 1 (under the assumption $n \geqslant 2$) we get the constant zero sequence (because of the numerator), so it converges to 0, and not 1, so it is not continuous.

   b) $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{, for } (x,y) \neq (0,0) \\ 0 & \text{, for } (x,y) = (0,0), \end{cases}$
   Except for $(0,0)$ it is continuous. At $(0,0)$ also, because
   $$\left|\frac{x^3+y^3}{x^2+y^2}\right| = \frac{|x^3+y^3|}{r^2} \leqslant \frac{|x^3|+|y^3|}{r^2} \leqslant \frac{2r^3}{r^2} = 2r \to 0.$$
   Thus, the function is continuous.

   c) $f(x,y) = \begin{cases} \dfrac{2xy^3+x^2y^3}{x^4+2y^4} & \text{, for } (x,y) \neq (0,0) \\ 0 & \text{, for } (x,y) = (0,0), \end{cases}$

It is continuous except for $(0,0)$, and not continuous there, since for $x_n = y_n = 1/n$ we get:

$$\frac{2xy^3 + x^2y^3}{x^4 + 2y^4} = \frac{2/n^4 + 1/n^5}{3/n^4} = 2/3 + 1/3n,$$

which converges to $2/3$, and the value of the function is 0.

3. Prove that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{, for } (x,y) \neq (0,0) \\ 0 & \text{, for } (x,y) = (0,0), \end{cases}$$

is not continuous at $(0,0)$.

In is enough to take $x_n = y_n = 1/n$ and we get

$$\frac{xy}{x^2+y^2} = \frac{1/n^2}{2/n^2} = 1/2,$$

but the value at $(0,0)$ is 0.