Mathematical analysis 2, WNE, 2018/2019
meeting 8. – solutions

14 March 2019

1. Find the domain of the function
$$f(x,y)=\sqrt{\frac{x}{x^2+y^2+2x}-1}.$$
A point (x,y) is in the domain, if $\frac{x}{x^2+y^2+2x}-1\geqslant 0$, so $\frac{x^2+y^2+x}{x^2+y^2+2x}\leqslant 0$, which is possible only if $x<0$ and $x^2+y^2+2x<0$ i $x^2+y^2+x\geqslant 0$. I.e. if
$$(x+1)^2+y^2<1$$
and in the same time
$$\left(x+\frac{1}{2}\right)^2+y^2\geqslant\frac{1}{4},$$
These are points in the circle of radius 1 and centre in $(-1,0)$ except for the circle of radius $1/2$ and centre in $(-1/2,0)$. So it is $B((-1,0),1)\setminus B((-1/2,0),1/2)$.

2. Does
$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^4-y^4}{x+y}$$
exist? If so calculate it.
We have $\frac{x^4-y^4}{x+y}=(x-y)(x^2+y^2)$. Let $x_n\to 0$ and $y_n\to 0$ be such that $x_n+y_n\neq 0$. Then
$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^4-y^4}{x+y}=\lim_{n\to\infty}\frac{x_n^4-y_n^4}{x_n+y_n}=\lim_{n\to\infty}(x_n-y_n)(x_n^2+y_n^2)=0.$$

3. Find whether the following functions is continuous in each of the points if its domain.
$$f(x,y)=\begin{cases}1 & \text{, for } xy>0,\\ 0 & \text{, for } xy=0,\\ -1 & \text{, for } xy<0.\end{cases}$$
It is clear that this function is continuous everywhere except the axis (i.e. for $xy>0$ and $xy<0$), and the axis are suspicious (i.e. $x=0\vee y=0$). In these points the function is not continuous. If (x,y) is such that $x=0$ or $y=0$, then let $x_n=x+\frac{x}{n}+\frac{y}{n}$, $y=y+\frac{1}{n}+\frac{x}{n}+\frac{y}{n}$. Then $\lim_{n\to\infty}f(x_n,y_n)=1$ or $\lim_{n\to\infty}f(x_n,y_n)=-1$, but $f(x,y)=0$.

4. Let
$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2} & \text{, for } (x,y)\neq(0,0)\\ 0 & \text{, for } x=y=0.\end{cases}$$
Prove that despite the limit in $(0,0)$ if you approach it by any line going through $(0,0)$, f is not continuous in $(0,0)$.
For $y=ax$ we have
$$\frac{x^2y}{x^4+y^2}=\frac{ax^3}{(x^2+a^2)x^2}=\frac{ax}{x^2+a^2}\to 0,$$

when $x \to 0$. But if $x = 1/n, y = 1/n^2$ we get

$$\frac{x^2 y}{x^4 + y^2} = \frac{1/n^4}{1/n^4 + 1/n^4} = \frac{1}{2}.$$

so the function has no limit at $(0,0)$, thus it is not continuous.

5. Let $f\colon \mathbb{R}^2 \setminus \{(0,0)\}$ be defined as

$$f(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2},$$

Prove that despite

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 0 = \lim_{y\to 0}\lim_{x\to 0} f(x,y),$$

the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Numerator converges to 0 for $x \to 0$ and constant y, and the denominator is non-zero so

$$\lim_{y\to 0} \frac{x^2y^2}{x^2y^2 + (x-y)^2} = 0,$$

so

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 0.$$

Similarly, if we take the limits in the other order.

But for $x = y = 1/n$ we get

$$\frac{x^2y^2}{x^2y^2 + (x-y)^2} = \frac{1/n^4}{1/n^4 + 0} = 1,$$

so the limit here is 1, thus the limit of the function does not exist.

6. Prove that a function $f\colon \mathbb{R}^k \to \mathbb{R}$ is continuous if and only if for any open set $G \subseteq \mathbb{R}$ the set

$$\{(x_1,\ldots,x_k) \in \mathbb{R}^k \colon f(x_1,\ldots,x_k) \in G\}$$

is open in $\mathbb{R}^k$.

Assume that f is continuous, and $G \subseteq \mathbb{R}$ is open. Assume that $(x_1,\ldots,x_k)$ is such that $g = f(x_1,\ldots,x_k) \in G$. But since G is open, there exists $\varepsilon > 0$, such that $(g-\varepsilon, g+\varepsilon) \subseteq G$. But since f is continuous, there exists $r > 0$, such that if $\|(x_1,\ldots,x_k) - (y_1,\ldots,y_k)\| < r$, to $f(y_1,\ldots,y_k) \in (g-\varepsilon, g+\varepsilon) \subseteq G$, and also a ball with centre in $(x_1,\ldots,x_k)$ and radius r is contained in

$$\{(x_1,\ldots,x_k) \in \mathbb{R}^k \colon f(x_1,\ldots,x_k) \in G\}$$

So it is open.

Reversely, fix $(x_1,\ldots,x_k)$ and $\varepsilon > 0$, and let $g = f(x_1,\ldots,x_k)$ and $G = (g-\varepsilon, g+\varepsilon)$. Then

$$\{(x_1,\ldots,x_k) \in \mathbb{R}^k \colon f(x_1,\ldots,x_k) \in G\}$$

is open, so there exists $r > 0$, such that the ball with centre in $(x_1,\ldots,x_k)$ and radius r is included in this set. But ε is arbitrary, so we conclude that the function is continuous.