# Mathematical Analysis 2, WNE, 2018/2019
# meeting 7. – solutions

12 March 2019

1. Check whether the following sets are bounded, open, closed, compact or convex:

   a) $\{(x, y) \in \mathbb{R}^2 : |x| \leqslant 1 \wedge |y| < 2\}$,
   It is a rectangle without its left and right side, but with its upper and bottom sides.
   It is bounded and convex, but neither open nor closed nor compact.

   b) $\{(x, y, z) \in \mathbb{R}^3 : 2x + y - 3z \leqslant 7\}$,
   It is a halfspace under the plane $2x + y - 3z = 7$. The plane is included in the set.
   It is closed and convex, but neither open nor bounded nor compact.

   c) $\{(x, y) \in \mathbb{R}^2 : x^2 + 9y^2 > 1\}$,
   It is outside of an ellipse without its border.
   It is open, but unbounded, neither closed nor convex nor compact.

   d) $\{(x, y) \in \mathbb{R}^2 : x + y^2 < 1\}$,
   The area on one side (inner) of a parabola without the parabola itself.
   It is open and convex, but neither bounded nor closed nor compact.

   e) $\{(x, y, z) \in \mathbb{R}^3 : 4 \leqslant x^2 + y^2 + z^2 < 9\}$,
   A ball with a smaller ball cut out in the middle. The inner surface is included, the outer is not.
   It is bounded, but neither convex nor open nor closed nor compact.

   f) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z \geqslant 1 \wedge x^2 + y^2 + z^2 < 4\}$.
   The intersection of the outside of parabola and a ball. The surface of the parabola is included. The surface of the ball is not.
   It is bounded, but neither convex nor open nor closed nor compact.

2. Check whether the limit exists. If so, calculate it. Otherwise explain why it does not exist.

   a) $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$,
   There is no limit because for $x = 0$, $y = 1/n$ we get
   $$\frac{xy}{x^2 + y^2} = 0 \to 0,$$
   but for $x = 1/n$, $y = 1/n$ we get
   $$\frac{xy}{x^2 + y^2} = \frac{1/n^2}{2/n^2} = \frac{1}{2} \to 1/2.$$

   b) $\lim_{(x,y)\to(1,1)} \dfrac{y^3 - x^3}{x - y}$,
   We get
   $$\frac{y^3 - x^3 + 3x^2y - 3xy^2}{x - y} = \frac{(y - x)^3}{x - y} = -(y - x) \to 0.$$
   and
   $$\frac{3x^2y - 3xy^2}{x - y} = \frac{3xy(x - y)}{x - y} = 3xy \to 3.$$
   So
   $$\frac{y^3 - x^3}{x - y} = \frac{y^3 - x^3 + 3x^2y - 3xy^2}{x - y} - \frac{3x^2y - 3xy^2}{x - y} \to 0 - 3 = -3.$$

c) $\lim_{(x,y)\to(0,0)} \dfrac{\sin xy}{x^2+y^2}$,

There is no limit because for $x = 0$, $y = 1/n$ we get

$$\frac{\sin xy}{x^2+y^2} = 0 \to 0,$$

but for $x = 1/n$, $y = 1/n$ we get

$$\frac{\sin xy}{x^2+y^2} = \frac{\sin 1/n^2}{2/n^2} = \frac{1}{2}\cdot\frac{\sin 1/n^2}{1/n^2} \to 1/2\cdot 1 = 1/2.$$

d) $\lim_{(x,y)\to(0,0)} \dfrac{\sqrt{1+x^2+y^2}-1}{x^2+y^2}$,

$x^2+y^2 = r^2$, so

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{1+x^2+y^2}-1}{x^2+y^2} = \lim_{r\to 0}\frac{\sqrt{1+r^2}-1}{r^2} = \lim_{r\to 0}\frac{1+r^2-1}{r^2(\sqrt{1+r^2}+1)} = \lim_{r\to 0}\frac{1}{\sqrt{1+r^2}+1} = 1/2.$$

e) $\lim_{(x,y)\to(0,1)} \dfrac{\sin xy}{x}$,

We get $\dfrac{\sin xy}{xy} \to 1$, because $xy \to 0$. Thus, $\frac{\sin xy}{x} = y\cdot\frac{\sin xy}{xy} = 1\cdot 1 = 1$.

f) $\lim_{(x,y)\to(0,0)} \dfrac{\sin xy}{x}$,

We get $\dfrac{\sin xy}{xy} \to 1$, because $xy \to 0$. Thus, $\frac{\sin xy}{x} = y\cdot\frac{\sin xy}{xy} = 0\cdot 1 = 0$.

g) $\lim_{(x,y)\to(0,0)} \dfrac{x}{x^2+y^2}$,

There is no limit. Indeed, if $x = 1/n$, $y = 0$, then $\frac{x}{x^2+y^2} = \frac{1/n}{1/n^2} = n \to \infty$, but if $x = -1/n$, $y = 0$, then $\frac{x}{x^2+y^2} = \frac{-1/n}{1/n^2} = -n \to -\infty$.

h) $\lim_{(x,y)\to(0,0)} y\ln(x^2+y^2)$.

$x^2+y^2 = r^2 \to 0$, i $|y| \leqslant r$, so

$$|y\ln(x^2+y^2)| = |y||\ln r^2| \leqslant r|\ln r^2| = \frac{-\ln r^2}{1/r},$$

has the same limit as

$$\frac{-2r\cdot 1/r^2}{-1/r^2} = 2r \to 0,$$

so the limit of the function in question is 0.