Analiza matematyczna 2, WNE, 2018/2019
ćwiczenia 6. – rozwiązania

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1. Prove that every norm generated by an inner product satisfies (the parallelogram law)

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2),$$

for every points $u, v \in \mathbb{R}^n$.

$$\|u+v\|^2 + \|u-v\|^2 = \langle u+v, u+v\rangle + \langle u-v, u-v\rangle = 2\langle u,u\rangle + 2\langle u,v\rangle - 2\langle u,v\rangle + 2\langle v,v\rangle =$$
$$= 2\langle u,u\rangle + 2\langle v,v\rangle = 2(\|u\|^2 + \|v\|^2).$$

2. Prove that every inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$ the norm generated by it satisfy the following condition:

$$\langle u,v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2),$$

for every points $u, v \in \mathbb{R}^n$.

$$\|u+v\|^2 - \|u-v\|^2 = \langle u+v, u+v\rangle - \langle u-v, u-v\rangle = 2\langle u,v\rangle + 2\langle u,v\rangle = 4\langle u,v\rangle.$$

3. Prove the Jordan-von Neumann Theorem, which states that every norm satisfying the parallelogram law is generated by an inner product (hint: the previous problem).

Let

$$\langle u,v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2),$$

obviously the function is symmetrical with regard to u and v.

We also get

$$\langle u,u\rangle = \frac{1}{4}(\|u+u\|^2 - \|u-u\|^2) = 4\|u\|^2/4 = \|u\|^2 \geqslant 0,$$

and is zero only if $u = 0$.

We have

$$\|x+z+y\|^2 + \|x+z-y\|^2 = 2(\|x+z\|^2 + \|y\|^2)$$

and

$$\|x-z+y\|^2 + \|x-z-y\|^2 = 2(\|x-z\|^2 + \|y\|^2)$$

subtracting these we get

$$-\|x-z+y\|^2 - \|x-z-y\|^2 + \|x+z+y\|^2 + \|x+z-y\|^2 = 2(-\|x-z\|^2 + \|x+z\|^2)$$

so (by definition of $\langle\cdot,\cdot\rangle$),

$$\langle x+y,z\rangle + \langle x-y,z\rangle = 2\langle x,z\rangle.$$

Substituting $x = (u+v)/2$, $y = (u-v)/2$ and $w = z$ we immediately get

$$\langle u,w\rangle + \langle v,w\rangle = \langle u+v,w\rangle.$$

Applying it multiple times we get $a \in \mathbb{Z}$,

$$\langle au,v\rangle = a\langle u,v\rangle,$$

and applying it inversely we get

$$\langle au, v\rangle = a\langle u, v\rangle,$$

for $a \in \mathbb{Q}$. For any u, v, let $f(a) = a\langle u, v\rangle$ and $g(a) = \langle au, v\rangle$. Both those functions are continuous and they are equal on rationals so they have to be equal for any $a \in \mathbb{R}$.

Then $\|u\| = \sqrt{4\|u\|^2/4} = \sqrt{\frac{1}{4}(\|u+u\|^2 - \|u-u\|^2)} = \sqrt{\langle u, u\rangle}$.

4. Prove that the unit ball for every norm is convex.

   Let $x, y \in B$, where B where B is the unit ball. Then for any $t \in [0, 1]$ we have

$$\|x + t(y-x)\| = \|(1-t)x + ty\| \leqslant \|(1-t)x\| + \|ty\| = (1-t)\|x\| + t\|y\| \leqslant (1-t) + t = 1.$$

5. Let $W \subseteq \mathbb{R}^n$ be a convex set such that:

   a) for every $v \in \mathbb{R}^n$, there exists $t \in \mathbb{R}$, such that $v \in tW = \{tw\colon w \in W\}$,

   b) for every $w \in W$ and $r \in [-1, 1]$, $rw \in W$,

   c) there exists $R > 0$, such that for every $(w_1, \ldots, w_n) \in W$, $w_1^2 + \ldots + w_n^2 \leqslant R$.

   Prove that

$$\|v\| = \inf\{t > 0\colon tv \in W\}$$

   is a norm in $\mathbb{R}^n$.

   Indeed, if $\|v\| = 0$, to $v \in \{0\}$ (by the third condition), so $v = 0$.

   We also have $tav \in W$ is and only if $t|a|v \in W$, so $\|av\| = |a|\|v\|$.

   If $u \in tW$ and $v \in tW$, then $u + v = 2u/2 + 2v/2 \in 2tW$ because W is convex. Thus, $\|u+v\| \leqslant \|u\| + \|v\|$.