# Mathematical analysis 2, WNE, 2018/2019
# meeting 5. – solutions

5 March 2019

1. Using the triangle inequality prove that
$$|\|x\| - \|y\|| \leqslant \|x - y\|,$$
for any norm $\| \cdot \|$ and points $x, y \in \mathbb{R}^n$.

   Without a loss of generality, assume that $\|x\| \geqslant \|y\|$. Then $|\|x\| - \|y\|| = \|x\| - \|y\|$, but for $z = x - y$ from the triangle inequality we get $\|z\| + \|y\| \geqslant \|z + y\| = \|x - y + y\| = \|x\|$, and thus $\|z\| \geqslant \|x\| - \|y\|$, so $|\|x\| - \|y\|| = \|x\| - \|y\| \leqslant \|z\| = \|x - y\|$.

2. Prove that the Euclidean norm in $\mathbb{R}^2$ is actually a norm.

   Indeed, $\sqrt{x^2 + y^2} = 0$ if and only if $x^2 + y^2 = 0$, if and only if $x = y = 0$. Moreover, for any $\alpha$, $\sqrt{\alpha^2 x^2 + \alpha^2 y^2} = |\alpha|\sqrt{x^2 + y^2}$. Finally, for $v = (x, y)$, $w = (x', y')$ we have
$$(x + x')^2 + (y + y')^2 = x^2 + y^2 + x'^2 + y'^2 + 2xx' + 2yy' \leqslant x^2 + y^2 + x'^2 + y'^2 + 2|xx'| + 2|yy'| \leqslant$$
$$\leqslant x^2 + y^2 + x'^2 + y'^2 + 2\sqrt{(x^2 + y^2)(x'^2 + y'^2)} = (\sqrt{x^2 + y^2} + \sqrt{x'^2 + y'^2})^2,$$
since
$$(xx')^2 + (yy')^2 + 2(xx'yy') \leqslant x^2x'^2 + y^2y'^2 + x^2y'^2 + y^2x'^2,$$
since
$$(xy' - yx')^2 \geqslant 0.$$

3. Prove that the Euclidean norm in $\mathbb{R}^n$ is actually a norm. Hint: use the Schwartz inequality.

   Schwartz inequality states that
$$(x_1x_1' + \ldots + x_nx_n')^2 \leqslant (x_1^2 + \ldots + x_n^2)((x_1')^2 + \ldots + (x_n')^2).$$
Similarly as in the case of $\mathbb{R}^2$, the first two properties of the norm are obvious. Triangle inequality:
$$(x_1 + x_1')^2 + \ldots + (x_n + x_n')^2 = x_1^2 + \ldots + x_n^2 + (x_1')^2 + \ldots + (x_n')^2 + 2|x_1x_1' + \ldots + x_nx_n'| \leqslant$$
$$\leqslant x_1^2 + \ldots + x_n^2 + (x_1')^2 + \ldots + (x_n')^2 + 2\sqrt{(x_1^2 + \ldots + x_n^2)((x_1')^2 + \ldots + (x_n')^2)} =$$
$$= \left(\sqrt{x_1^2 + \ldots + x_n^2} + \sqrt{(x_1')^2 + \ldots + (x_n')^2}\right)^2.$$

4. Show that any inner product in $\mathbb{R}^n$ defines a norm $\|v\|$ given by $\sqrt{\langle v, v\rangle}$.

   $\sqrt{\langle v, v\rangle} = 0$ if and only if $\langle v, v\rangle = 0$ if and only if $v = 0$.
   $\sqrt{\langle av, av\rangle} = \sqrt{a\,\langle v, av\rangle} = \sqrt{a\,\langle av, v\rangle} = \sqrt{a^2\,\langle v, v\rangle} = |a|\sqrt{\langle v, v\rangle}$.
   The triangle inequality follows from the Schwartz inequality. $\|u + v\|^2 = \langle u + v, u + v\rangle = \|u\|^2 + \|v\|^2 + 2\,\langle u, v\rangle \leqslant \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| = (\|u\| + \|v\|)^2$.