Mathematical analysis 2, WNE, 2018/2019
meeting 11. – solutions

26 March 2019

1. Find all functions f, such that
$$\frac{\partial f}{\partial x}(x,y) = 2xy^3 + e^x \sin y,$$
$$\frac{\partial f}{\partial y}(x,y) = 3x^2y^2 + e^x \cos y + 1.$$
Obviously,
$$f(x,y) = x^2y^3 + e^x \sin y + y + C.$$

2. Find the directional derivative of the above function at point $(0,0)$ in direction $v = (2,1)$.
$$\frac{\partial f}{\partial} v(0,0) = \lim_{h\to 0} \frac{f((0,0)+hv) - f(0,0)}{h} = \lim_{h\to 0} \frac{4h^5 + e^{2h}\sin h + h + C - C}{h} = 0 + 2\cdot 1 + 1 = 3.$$

3. Examine the differentiability of:

   a) $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} \text{ , for } (x,y)\neq(0,0) \\ 0 \text{ , for } (x,y) = (0,0) \end{cases}$,

   Obviously apart from $(0,0)$ there are no problems. We first check partial derivatives at $(0,0)$.
$$\frac{\partial}{f}\partial x(0,0) = \lim_{h\to 0} \frac{0(0+h)\frac{(0+h)^2-0^2}{(0+h^2)+0^2} - 0}{h} = 0,$$
$$\frac{\partial}{f}\partial x(0,0) = \lim_{h\to 0} \frac{(0+h)0\frac{0^2-(0+h)^2}{0^2+(0+h^2)} - 0}{h} = 0,$$
   So the candidate for the derivative is $A(h_x,h_y) = 0$. We check the limit
$$\lim_{(h_x,h_y)\to(0,0)} \frac{|f((0,0)+(h_x,h_y)) - f(0,0) - A(h_x,h_y)|}{\|(h_x,h_y)\|} =$$
$$= \lim_{(h_x,h_y)\to(0,0)} \frac{|h_xh_y\dfrac{h_x^2-h_y^2}{h_x^2+h_y^2}|}{\sqrt{h_x^2+h_y^2}} = \lim_{(h_x,h_y)\to(0,0)} \left| h_xh_y\frac{h_x^2-h_y^2}{(h_x^2+h_y^2)^{3/2}} \right| = 0,$$
   since
$$\lim_{(h_x,h_y)\to(0,0)} \left| h_xh_y\frac{h_x^2-h_y^2}{(h_x^2+h_y^2)^{3/2}} \right| \leqslant \lim_{r\to 0}\frac{r^4}{r^3} = 0.$$
   So the derivative exists and $Df_{(0,0)}(h_x,h_y) = 0$.

   b) $f(x_1,\ldots,x_k) = \sqrt{x_1^2+\ldots+x_k^2}$,

   Obviously apart from $(0,0)$ there are no problems. We first check partial derivatives at $(0,0)$.
$$\frac{\partial}{f}\partial x_i(0,0) = \lim_{h\to 0} \frac{\sqrt{0+\ldots+h^2+\ldots+0} - 0}{h} = 1,$$

So the candidate is $A(h_1,\dots,h_k)=h_1+\dots+h_k$. We check the limit

$$\lim_{(h_1,\dots,h_k)\to(0,\dots,0)} \frac{\sqrt{h_1^2+\dots+h_k^2}-0-(h_1+\dots+h_k)}{\sqrt{h_1^2+\dots+h_k^2}},$$

but the limit does not exist $h_1=1/n$, $h_2=\dots=h_k=0$ we get 0, but for $h_1=1/n$, $h_2=-1/n$, $h_3=\dots=h_k=0$, we get 1.
So the function is not differentiable at $(0,0)$.

c) $f(x,y)=\begin{cases}\dfrac{\sin xy}{y}\ , \text{ for } y\neq 0\\ x\ , \text{ for } y=0\end{cases}$.

Here we need to check what happens for (x_0,y). We first check partial derivatives.

$$\frac{\partial f}{\partial x}f(x_0,0)=\lim_{h\to 0}\frac{x_0+h-x_0}{h}=1,$$

$$\frac{\partial f}{\partial y}f(x_0,0)=\lim_{h\to 0}\frac{\frac{\sin(hx_0)}{h}-x_0}{h}=x_0-0=x_0.$$

Thus the candidate for the derivative is $A(h_x,h_y)=h_x+x_0h_y$. We check the limit.

$$\lim_{(h_x,h_y)\to(0,0)}\frac{|f((x_0,0)+(h_x,h_y))-f(x_0,0)-A(h_x,h_y)|}{\|(h_x,h_y)\|}=$$

$$=\lim_{(h_x,h_y)\to(0,0)}\frac{\left|\frac{\sin(x_0+h_x)h_y}{h_y}-x_0-h_x-x_0h_y\right|}{\sqrt{h_x^2+h_y^2}}$$

It does not exist for $x_0\neq 0$, but for $h_x=h_y=h\to 0$ we have

$$\lim_{h\to 0}\frac{\frac{\sin(hx_0+h^2)}{h}-x_0-h-x_0h}{\sqrt{2}h}=\frac{1}{\sqrt{2}}\lim_{h\to 0}\left(\frac{\sin(hx_0+h^2)-hx_0}{h^2}-1-x_0\right)=\infty,$$

if $x_0\neq 0$. For $x_0=0$ we have

$$\lim_{(h_x,h_y)\to(0,0)}\frac{|\sin h_xh_y-h_xh_y|}{h_y\sqrt{h_x^2+h_y^2}}=\lim_{(h_x,h_y)\to(0,0)}\left|\frac{sinh_xh_y-h_xh_y}{h_xh_y}\cdot\frac{h_x}{\sqrt{h_x^2+h_y^2}}\right|=0,$$

as a product of a function converging to zero and a bounded function. So it is differentiable at $(0,0)$, but is not at $(x_0,0)$ for $x_0\neq 0$.

d) $f(x,y,z)=\sqrt{xyz}$

As usually, the problem may be $(0,0,0)$. We check partial derivatives

$$\frac{\partial f}{\partial x}(0,0,0)=\lim_{h\to 0}\frac{0-0}{h}=0$$

and analogously for y i z. So the candidate for the derivative is $A(h_x,h_y,h_z)=0$, but

$$\lim_{(h_x,h_y,h_z)\to(0,0,0)}\frac{|\sqrt{h_xh_yh_z}-0-0|}{\sqrt{h_x^2+h_y^2+h_z^2}}$$

does not exist. Actually, for $h_x=h_y=h_z=1/n$ the sequence goes to infinity.

4. Calculate partial derivatives of $g\circ f$:

a) $f\colon\mathbb{R}^+\to\mathbb{R}^2$, $f(x)=(x,\sqrt{x})$, $g\colon\mathbb{R}^2\to\mathbb{R}$, $g(a,b)=e^{-(a^2+b^2)}$,

First method: $g(f(x))=e^{-(x^2+x)}$, so $(g\circ f)'(x)=-(2x+1)e^{-(x^2+x)}$.

Second method:

$$f'(x) = \begin{bmatrix} 1 \\ \frac{1}{2\sqrt{x}} \end{bmatrix}.$$

$$g'(a,b) = \left[-2ae^{-(a^2+b^2)}, -2be^{-(a^2+b^2)}\right]$$

$$g'(f(x)) = \left[-2xe^{-(x^2+x)}, -2\sqrt{x}e^{-(x^2+x)}\right]$$

Thus,

$$(g \circ f)' = g'(f(x))) \cdot f'(x) = -2xe^{-(x^2+x)} - \frac{2\sqrt{x}e^{-(x^2+x)}}{2\sqrt{x}} = e^{-(a^2+b^2)}.$$

b) $f\colon \mathbb{R} \to \mathbb{R}^2$, $f(x) = (\cos x, \sin x)$, $g\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$, $g(a,b) = \dfrac{1}{a^2+b^2}$,

First method:

$$g(f(x)) = \frac{1}{\cos^2 x + \sin^2 x} = 1,$$

so $(g \circ f)'(x) = 0$.

c) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x-y, x+y)$, $g\colon \mathbb{R}^2 \to \mathbb{R}^2$, $g(a,b) = (e^a \cos b, e^a \sin b)$.

Second method:

$$f'(x,y) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

$$g'(a,b) = \begin{bmatrix} e^a \cos b & -e^a \sin b \\ e^a \sin b & e^a \cos b \end{bmatrix},$$

$$g'(f(x,y)) = \begin{bmatrix} e^{x-y} \cos b & -e^{x-y} \sin(x+y) \\ e^{x-y} \sin(x+y) & e^{x-y} \cos(x+y) \end{bmatrix},$$

$$(g \circ f)' = g'(f(x,y))) \cdot f'(x,y) =$$

$$= \begin{bmatrix} e^{x-y} \cos b - e^{x-y} \sin(x+y) & -e^{x-y} \cos b - e^{x-y} \sin(x+y) \\ e^{x-y} \sin(x+y) + e^{x-y} \cos(x+y) & -e^{x-y} \sin(x+y) + e^{x-y} \cos(x+y) \end{bmatrix}.$$