Mathematical analysis 2, WNE, 2018/2019
meeting 4. – solutions

28 February 2019

1. Calculate the derivatives of the following functions:

   a) $F(x) = \int_1^x t(1+t^2)^5\,dt$,
   Calculating everything we get

   $$F(x) = \int_1^x t(1+t^2)^5\,dt \overset{\boxed{u=1+t^2,\ \frac{du}{dt}=2t}}{=} \int_2^{1+x^2} \frac{1}{2}u^5\,du = \left(\frac{u^6}{12}\right)|_1^{1+x^2}\ \frac{(1+x^2)^6-64}{12}.$$

   $$F'(x) = x(1+x^2)^5.$$

   But we can do this without calculation but using the main theorem on definite integrals $F(x) = G(x) - G(1)$, where $G'(t) = t(1+t^2)^5$, so $F'(x) = (G(x) - G(1))' = G'(x) = x(1+x^2)^5$.

   b) $F(x) = \int_0^{x^2} t\sin t\,dt$,
   $F(x) = G(x^2) - G(0)$, where $G'(t) = t\sin t$, so $F'(x) = (G(x^2) - G(0))' = 2xG'(x^2) = 2x^3\sin x^2$.

   c) $F(x) = \int_1^{1+x^2} \sqrt{1+t}\,dt$,
   $F(x) = G(1+x^2) - G(1)$, where $G'(t) = \sqrt{1+t}$, so $F'(x) = (G(1+x^2) - G(1))' = 2xG'(1+x^2) = 2x\sqrt{2+x^2}$.

   d) $F(x) = \int_x^{x^2} t^{-2}\,dt$
   $F(x) = G(x^2) - G(x)$, where $G'(t) = t^{-2}$, so $F'(x) = (G(x^2) - G(x))' = 2xG'(x^2) - G'(x) = 2/x - 1/x^2$.

2. Calculate:

   $$\frac{d}{da}\int_a^b \sin^2 x\,dx.$$

   $F(a) = G(b) - G(a)$, where $G'(x) = \sin^2 x\,dx$, so $F'(a) = (G(b) - G(a))' = -G'(a) = -\sin^2 a$.

   I we want to calculate this through, first we need $\int \sin^2 x\,dx$. By parts:

   $$\int \sin^2 x\,dx = \int \sin x \cdot \sin x\,dx = -\sin x\cos x + \int \cos^2 x\,dx =$$

   $$= -\sin x\cos x + \int dx - \int \sin^2 x\,dx = -\sin x\cos x + x - \int \sin^2 x\,dx.$$

   Thus,

   $$\int \sin^2 x\,dx = \frac{-\sin x\cos x + x}{2}.$$

   So,

   $$\frac{d}{da}\int_a^b \sin^2 x\,dx = \frac{d}{da}\left(\frac{-\sin b\cos b + b}{2} - \frac{-\sin a\cos a + a}{2}\right) = 0 - \frac{-\cos^2 a + \sin^2 a + 1}{2} = -\sin^2 a.$$

3. Calculate:

a) $\int_{-2}^{0} \frac{dx}{\sqrt[3]{x}}$,

$$\int_{-2}^{0} \frac{dx}{\sqrt[3]{x}} = \lim_{t\to 0} \int_{-2}^{t} \frac{dx}{\sqrt[3]{x}} = \lim_{t\to 0} \left(\frac{3x^{2/3}}{2}\right)|_{-2}^{t} = \lim_{t\to 0} \left(\frac{3t^{2/3}}{2} + \frac{3\sqrt[3]{4}}{2}\right) = \frac{3\sqrt[3]{4}}{2}$$

b) $\int_0^\infty e^{-x}\,dx$,

$$\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_0^t e^{-x}\,dx = \lim_{t\to\infty} -e^{-t} + 1 = 1.$$

c) $\int_0^\infty xe^x\,dx$,

$$\int_0^\infty xe^x\,dx = \lim_{t\to\infty} \int_0^t xe^x = \lim_{t\to\infty} (xe^x - e^x)|_0^t = \lim_{t\to\infty} e^t(t-1) + 1 = \infty$$

d) $\int_2^\infty \frac{dx}{x^2+x-2}$,

$$\int_0^\infty \frac{dx}{x^2+x-2} = \lim_{t\to\infty} \int_0^t \left(-\frac{1/3}{x+2} + \frac{1/3}{x-1}\right) dt = \frac{1}{3} \lim_{t\to\infty} \ln(t-1) - \ln(t+2) - 0 + \ln 2 = \frac{\ln 2}{3}.$$

e) $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$.

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 0} \int_{-1}^{t} \frac{dx}{\sqrt{1-x^2}} + \lim_{t\to 0} \int_{t}^{1} \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 0} \arcsin x|_{-1}^{t} + \lim_{t\to 0} \arcsin x|_{t}^{1} = \pi/2 + \pi/2 = \pi.$$