Mathematical analysis 2, WNE, 2018/2019
meeting 3. – solutions

26 February 2019

1. Using the definition of the definite Riemann's integral calculate:

$$\int_1^2 \frac{dx}{x^2}.$$

The integral exists because the function if continuous. Thus, we can start with any sequence of intervals, e.g. $\{(1+k/n, 1+(k+1)/n)\colon k \in \{0,\ldots n-1\}\}$. In the interval, $(1+k/n, 1+(k+1)/n)$ we choose point

$$\sqrt{(1+k/n)(1+(k+1)/n)}.$$

$$\int_1^2 \frac{dx}{x^2} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{n} \cdot \frac{1}{(1+k/n)(1+(k+1)/n)} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{1+k/n} - \sum_{k=0}^{n-1} \frac{1}{1+(k+1)/n} = \lim_{n\to\infty} 1 - \frac{1}{2} = \frac{1}{2}.$$

2. Calculate:

$$\int_0^\pi \sin x\, dx.$$

$$\int_0^\pi \sin x\, dx = -\cos x|_0^\pi = -\cos\pi - (-\cos 0) = 1 + 1 = 2.$$

3. Calculate:

a) $\int_0^5 |x^2-4|\, dx$,

$$\int_0^5 |x^2-4|\, dx = -\int_0^2 (x^2-4)\, dx + \int_2^5 (x^2-4)\, dx = -\left(\frac{x^3}{3} - 4x\right)|_0^2 + \left(\frac{x^3}{3} - 4x\right)|_2^5 =$$
$$= -\frac{8}{3} + 8 + \frac{125}{3} - 20 - \frac{8}{3} + 8 = \frac{97}{3}.$$

b) $\int_{e^{-2}}^{e^2} |\ln x|\, dx$,

$$\int_{e^{-2}}^{e^2} |\ln x|\, dx = -\int_{e^{-2}}^1 \ln x\, dx + \int_1^{e^2} |\ln x|\, dx =$$
$$= -x\ln x|_{e^{-2}}^1 + \int_{e^{-2}}^1 dx + x\ln x|_1^{e^2} - \int_1^{e^2} dx = -(0+2e^{-2}) + (1-e^{-2}) + (2e^2-0) - (e^2-1) = 2 - 3e^{-2} + e^2.$$

c) $\int_2^4 \frac{2\, dx}{2x-3}$,
We use substitution $t = 2x-3$, $\frac{dt}{dx} = 2, t(2) = 1, t(4) = 5$, so:

$$\int_2^4 \frac{2\, dx}{2x-3} = \int_1^5 \frac{dt}{t} = \ln|t||_1^5 = \ln 5 - 0 = \ln 5.$$

d) $\int_0^{2\pi} x|\cos x|dx$.

$$\int_0^{2\pi} x|\cos x|dx = \int_0^{\pi/2} x|\cos x|dx - \int_{\pi/2}^{3\pi/2} x|\cos x|dx + \int_{3\pi/2}^{2\pi} x|\cos x|dx =$$

$$= x\sin x|_0^{\pi/2} - \int_0^{\pi/2} \sin x\, dx - x\sin x|_{\pi/2}^{3\pi/2} + \int_{\pi/2}^{3\pi/2} \sin x\, dx + x\sin x|_{3\pi/2}^{2\pi} - \int_{3\pi/2}^{2\pi} \sin x\, dx =$$

$$= \frac{\pi}{2} + \cos x|_0^{\pi/2} + 2\pi - \cos x|_{\pi/2}^{3\pi/2} + \frac{3\pi}{2} + \cos x|_{3\pi/2}^{2\pi} = 4\pi - 1 + 1 = 4\pi.$$

4. Calculate:

a) $\int_1^\infty \frac{dx}{x^2}$,

$$\int_1^\infty \frac{dx}{x^2} = \lim_{\beta\to\infty} \int_1^\beta \frac{dx}{x^2} = \lim_{\beta\to\infty} -\frac{1}{x}|_1^\beta = -\lim_{\beta\to\infty} \frac{1}{\beta} - \frac{-1}{1} = 1.$$

b) $\int_0^\infty \frac{dx}{1+x^2}$.

$$\int_0^\infty \frac{dx}{x^2} = \lim_{\beta\to\infty} \int_0^\beta \frac{dx}{1+x^2} = \lim_{\beta\to\infty} -\mathrm{arctg} x|_0^\beta = \lim_{\beta\to\infty} \mathrm{arctg}\beta = \frac{\pi}{2}.$$

5. Calculate the area of the shape restricted by the axis X, the curve $y = \frac{1}{x^2}$ and lines $x = 1$ and $x = 2$.

$$P = \int_1^2 \frac{1}{x^2}\, dx = -\left.\frac{1}{x}\right|_1^2 = \frac{1}{2}.$$

6. Calculate the area of the shape restricted by $y = x^2$ and $y = x^3$.

They have an intersection for $x = 0$ and $x = 1$ and the second curve is on $[0, 1]$ below the first one, thus:

$$P = \int_0^1 (x^2 - x^3)\, dx = \left.\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\right|_0^1 = \frac{1}{12}.$$