Mathematical analysis 2, WNE, 2018/2019
meeting 2. – solutions

21 February 2019

Problems

1. Calculate:

   a) $\int e^{\sqrt{x}}\,dx$,
   We substitute $t=\sqrt{x}$, so $x=t^2$ and $dx=2t\,dt$. Then, by parts

$$\int e^{\sqrt{x}}\,dx = 2\int te^t\,dt = 2(te^t - \int e^t\,dt) = 2e^t(t-1)+C = 2e^{\sqrt{x}}(\sqrt{x}-1)+C.$$

   b) $\int \frac{\sqrt[6]{x}\,dx}{1+\sqrt[3]{x}}$.
   We substitute $t=\sqrt[6]{x}$, so

$$\int \frac{\sqrt[6]{x}\,dx}{1+\sqrt[3]{x}} = \int \frac{6t^7}{1+t^2}\,dt = \int (t^5-t^3+t)\,dt + \int \frac{t\,dt}{1+t^2} =$$

$$= t^6/6 - t^4/4 + t^2/2 + \frac{1}{2}\ln(t^2+1) + C = x/6 - t^{2/3}/4 + t^{1/3}/2 + \frac{1}{2}\ln(t^{1/3}+1)+C.$$

2. Calculate:

   a) $\int \frac{2\,dx}{(x-4)^4}$,

$$\int \frac{2\,dx}{(x-4)^4} \overset{\boxed{t=x-4,\ \frac{dt}{dx}=1}}{=} \int \frac{2\,dt}{t^4} = \frac{-2}{3t^3}+C = \frac{-2}{3(x-4)^3}+C.$$

   b) $\int \frac{2x+3}{(x^2+2x+4)^2}\,dx$,
   x^2+2x+4 has no roots, so we already have a simple fraction:

$$\int \frac{2x+3}{(x^2+2x+4)^2}\,dx = \int \frac{2x+2}{(x^2+2x+4)^2}\,dx + \int \frac{1}{(x^2+2x+4)^2}\,dx.$$

   For the first integral we come up with a simple substitution

$$\int \frac{2x+2}{(x^2+2x+4)^2}\,dx \overset{\boxed{t=x^2+2x+4,\ \frac{dt}{dx}=2x+2}}{=} \frac{-1}{x^2+2x+4}+C.$$

   The second integral needs a more complex substitution $t=\frac{x+p/2}{\sqrt{-\Delta/4}}$, in our case $\Delta=4-16=-12$ and $p=2$, so $t=\frac{x+1}{\sqrt{3}}$ i $\frac{dt}{dx}=\frac{1}{\sqrt{3}}$. Thus:

$$\int \frac{1}{(x^2+2x+4)^2}\,dx = \int \frac{\sqrt{3}\,dt}{9(t^2+1)^2} = \frac{\sqrt{3}}{9}\int \frac{dt}{(t^2+1)^2},$$

by the recurrent formula

$$\frac{\sqrt{3}}{9}\int\frac{dt}{(t^2+1)^2}=\frac{\sqrt{3}}{18}\left(\frac{t}{1+t^2}+\int\frac{dt}{1+t^2}\right)=\frac{\sqrt{3}}{18}\left(\frac{t}{1+t^2}+\operatorname{arctg}t\right)+C,$$

so finally

$$\int\frac{2x+3}{(x^2+2x+4)^2}\,dx=-\frac{1}{x^2+2x+4}+\frac{1}{6}\frac{x+1}{x^2+2x+4}+\frac{\sqrt{3}}{18}\operatorname{arctg}\left(\frac{x+1}{\sqrt{3}}\right)+C.$$

c) $\int\frac{dx}{x^4+x}$,
We have $(x^4+x)=x(x+1)(x^2-x+1)$ so we get simple fractions:

$$\frac{1}{x^4+x}=\frac{A}{x}+\frac{B}{x+1}+\frac{Cx+D}{x^2-x+1},$$

so

$$\begin{aligned}1&=A(x+1)(x^2-x+1)+Bx(x^2-x+1)+(Cx+D)x(x+1)=\\&=A(x^3+1)+B(x^3-x^2+x)+C(x^3+x^2)+D(x^2+x),\end{aligned}$$

thus:

$$\begin{cases}A=1\\B+D=0\\-B+C+D=0\\A+B+C=0\end{cases}.$$

Therefore: $A=1,B=-\frac{1}{3},C=-\frac{2}{3},D=\frac{1}{3}$, and:

$$\int\frac{dx}{x^4+x}=\int\frac{dx}{x}-\frac{1}{3}\int\frac{dx}{x+1}-\frac{1}{3}\int\frac{(2x-1)dx}{x^2-x+1}\overset{\boxed{t=x^2-x+1,\ \frac{dt}{dx}=2x-1}}{=}$$

$$=\ln|x|-\frac{1}{3}\ln|x+1|-\frac{1}{3}\ln|t|+C=\ln|x|-\frac{1}{3}\ln|x+1|-\frac{1}{3}\ln(x^2-x+1)+C=\ln\frac{|x|}{\sqrt[3]{|x+1|(x^2-x+1)}}+C.$$

d) $\int\frac{3x}{x^3-1}\,dx$,
We have: $x^3-1=(x-1)(x^2+x+1)$, so

$$\frac{3x}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1},$$

thus:

$$\begin{cases}A-C=0\\A-B+C=3\\A+B=0\end{cases},$$

and: $A=1,B=-1,C=1$ i

$$\int\frac{3x}{x^3-1}\,dx=\int\frac{dx}{x-1}+\int\frac{-x+1}{x^2+x+1}\,dx=$$

$$=\ln|x-1|-\frac{1}{2}\int\frac{2x+1}{x^2+x+1}\,dx+\frac{3}{2}\int\frac{dx}{x^2+x+1}=\ln|x-1|-\frac{\ln(x^2+x+1)}{2}+\frac{3}{2}\int\frac{dx}{x^2+x+1},$$

in the last case we need our substitution: $t=\frac{x+p/2}{\sqrt{-\Delta/4}}$, in our case $\Delta=1-4=-3$ and $p=1$, so $t=\frac{2x+1}{\sqrt{3}}$ i $\frac{dt}{dx}=\frac{2}{\sqrt{3}}$. Thus:

$$\int\frac{dx}{x^2+x+1}=\int\frac{\sqrt{3}/2}{\frac{3}{4}(t^2+1)}\,dt=\frac{2\sqrt{3}}{3}\operatorname{arctg}t=\frac{2\sqrt{3}}{3}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}+C,$$

so finally:

$$\int \frac{3x}{x^3-1}\,dx =$$
$$= \ln|x-1| - \frac{\ln(x^2+x+1)}{2} + \frac{3}{2}\frac{2\sqrt{3}}{3}\text{arctg}\frac{2x+1}{\sqrt{3}} + C.$$

e) $\int \frac{x^2-3x+2}{x(x^2+2x+1)}\,dx$
We have:

$$\int \frac{x^2-3x+2}{x(x^2+2x+1)}\,dx = \int \frac{x^2+2x+1-5x+1}{x(x^2+2x+1)}\,dx = \int \frac{dx}{x} - \int \frac{5x-1}{x(x^2+2x+1)}\,dx.$$

The second integral has to be decomposed

$$\frac{5x-1}{x(x^2+2x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+1}.$$

So

$$\begin{cases} A=-1 \\ 2A+C=5 \\ A+B=0 \end{cases},$$

thus $A=-1, B=1, C=7$ and

$$\int \frac{5x-1}{x(x^2+2x+1)}\,dx = -\int \frac{dx}{x} + \int \frac{x+7}{(x+1)^2}\,dx =$$
$$= -\ln|x| + \frac{1}{2}\int \frac{2x+2}{x^2+2x+1}\,dx + 6\int \frac{dx}{(x+1)^2} = -\ln|x| + \frac{\ln(x^2+2x+1)}{2} - \frac{6}{1+x} + C.$$

So, finally

$$\int \frac{x^2-3x+2}{x(x^2+2x+1)}\,dx = \frac{\ln(x^2+2x+1)}{2} - \frac{6}{1+x} + C.$$

3. Calculate:

a) $\int \frac{x+\sqrt{2x-3}}{x-1}\,dx.$

$$\int \frac{x+\sqrt{2x-3}}{x-1}\,dx \overset{\boxed{t=\sqrt{2x-3},\, x=\frac{t^2+3}{2},\, \frac{dt}{dx}=1/\sqrt{2x-3}}}{=}$$
$$= \int \frac{t^3+2t^2+3t}{t^2+1}\,dt = \int \left(t+2+\frac{2t}{t^2+1} - \frac{2}{t^2+1}\right)dt = \frac{t^2}{2} + 2t + \ln(t^2+1) - 2\text{arctg}t + C =$$
$$= \frac{2x-3}{2} + 2\sqrt{2x-3} + \ln(2x-2) - 2\text{arctg}\sqrt{2x-3} + C.$$

b) $\int \frac{dx}{e^x+e^{-x}},$

$$\int \frac{dx}{e^x+e^{-x}} \overset{\boxed{t=e^x,\, \frac{dt}{dx}=e^x}}{=} \int \frac{dt}{(1+t^{-1})t = \int \frac{dt}{t^2+1}} = \text{arctg}t + C = \text{arctg}e^x + C.$$

c) $\int \frac{dx}{3\sin x+\cos x}.$

$$\int \frac{dx}{3\sin x+\cos x} \overset{\boxed{t=\text{tg}\frac{x}{2},\, \frac{dt}{dx}=\frac{1+\text{tg}^2\frac{x}{2}}{2},\, \sin x=\frac{2t}{1+t^2},\, \cos x=\frac{1-t^2}{1+t^2}}}{=} \int \frac{\frac{2\,dt}{1+t^2}}{\frac{1}{1+t^2}(6t+1-t^2)} =$$

$$= \int \frac{2\,dt}{-t^2+6t+1} = \int \frac{dt}{t-3-\sqrt{10}} - \int \frac{dt}{t-3+\sqrt{10}} = \ln|t-3-\sqrt{10}| - \ln|t-3+\sqrt{10}| + C =$$
$$= \ln\left|\operatorname{tg}\frac{x}{2} - 3 - \sqrt{10}\right| - \ln\left|\operatorname{tg}\frac{x}{2} - 3 + \sqrt{10}\right| + C.$$