Mathematical analysis 2, WNE, 2018/2019
meeting 1. – solutions

19 February 2019

1. Let:
$$f(x) = \begin{cases} -x & x \in [0,1) \\ -1 & x \in [1,2) \\ x-3 & x \in [2,4] \end{cases}.$$
Check whether this function has an antiderivative. I so, find an antiderivative F, such that $F(1) = -1/2$.
The function f is continuous so it has an antiderivative. Thus,
$$F(x) = \begin{cases} -\frac{x^2}{2} + C & x \in [0,1) \\ -x + D & x \in [1,2) \\ \frac{x^2}{2} - 3x + E & x \in [2,4] \end{cases}$$
and $F(1) = \frac{-1}{2}$, so $D = \frac{1}{2}$. Moreover, it has to be continuous, so $C = 0$ and $E = \frac{5}{2}$, so finally:
$$F(x) = \begin{cases} -\frac{x^2}{2} & x \in [0,1) \\ -x + \frac{1}{2} & x \in [1,2) \\ \frac{x^2}{2} - 3x + \frac{5}{2} & x \in [2,4] \end{cases}.$$

2. Calculate:

a) $\int \frac{x^4 - 2x^3 + 4x^2 + x - 3}{x^2}\, dx$,
$$\int \frac{x^4 - 2x^3 + 4x^2 + x - 3}{x^2}\, dx = \int \left(x^2 - 2x + 4 + x^{-1} - 3x^{-2}\right) dx = \frac{x^3}{3} - x^2 + 4x + \ln|x| + 3x^{-1} + C.$$

b) $\int \frac{\sqrt{x} - x^3 e^x + x^2}{x^3}\, dx$.
$$\int \frac{\sqrt{x} - x^3 e^x + x^2}{x^3}\, dx = \int \left(x^{-\frac{5}{2}} - e^x + x^{-1}\right) dx = \frac{-2}{3x^{\frac{3}{2}}} - e^x + \ln|x| + C.$$

3. Calculate $\int \sin x \cos x \, dx$ using the following hints:

a) by parts for $f(x) = \sin x, g(x) = \cos x$,
$$\int \sin x \cos x \, dx = -\cos^2 x - \int \sin x \cos x \, dx,$$
so
$$\int \sin x \cos x \, dx = -\cos^2 x/2 + C.$$

b) by parts for $f(x) = \cos x, g(x) = \sin x$,
$$\int \sin x \cos x \, dx = \sin^2 x - \int \sin x \cos x \, dx,$$
so
$$\int \sin x \cos x \, dx = \sin^2 x/2 + C.$$

c) by substitution $y = \sin x$,

$$\int \sin x \cos x \, dx = \int y \, dy, = y^2/2 + C = \sin^2 x/2 + C.$$

d) by substitution $y = \cos x$,

$$\int \sin x \cos x \, dx = -\int y \, dy, = -y^2/2 + C = -\cos^2 x/2 + C.$$

e) using double angle formula $\sin x \cos x = \frac{1}{2}\sin(2x)$.

$$\int \sin x \cos x \, dx = \frac{1}{2}\int \sin(2x) \, dx = -\frac{1}{4}\cos(2x).$$

4. Calculate by parts:

a) $\int \ln|x| \, dx$,

$$\int \ln|x| \, dx \overset{\boxed{f(x)=x,\, g(x)=\ln|x|}}{=} \int f'(x)g(x) \, dx = f(x)g(x) - \int f(x)g'(x) \, dx =$$
$$= x\ln|x| - \int \frac{x}{x} \, dx = \ln|x| - \int 1 \, dx = x\ln|x| - x + C.$$

b) $\int x \cos x \, dx$,

$$\int x \cos x \, dx \overset{\boxed{f(x)=\sin x,\, g(x)=x}}{=} \int f'(x)g(x) \, dx = x\sin x - \int \sin x \cdot 1 \, dx = x \sin x + \cos x + C.$$

c) $\int x^2 e^{-x} \, dx$.
We use the method twice

$$\int x^2 e^{-x} \, dx \overset{\boxed{f(x)=-e^{-x},\, g(x)=x^2}}{=} \int f'(x)g(x) \, dx = -x^2 e^{-x} + 2\int x e^{x} \, dx \overset{\boxed{f(x)=-e^{-x},\, h(x)=x}}{=}$$
$$= -x^2 e^{-x} + 2\int f'(x)h(x) \, dx = -x^2 e^{-x} - 2xe^{-x} + 2\int e^{-x} \, dx =$$
$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C = -e^{-x}(x+1)^2 + C.$$

5. Calculate by substitution:

a) $\int x\sqrt{1+x^2} \, dx$,

$$\int x\sqrt{1+x^2} \, dx \overset{\boxed{t=1+x^2,\, \frac{dt}{dx}=2x}}{=} \int \frac{\sqrt{t}dt}{2} = \frac{1}{2}\cdot\frac{3}{2}t^{\frac{3}{2}} + C = \frac{\left(1+x^2\right)^{\frac{3}{2}}}{3} + C.$$

b) $\int x \cos x^2 \, dx$.

$$\int x \cos x^2 \, dx \overset{\boxed{t=x^2,\, \frac{dt}{dx}=2x}}{=} \int \frac{\cos t}{2} dt = \frac{\sin t}{2} + C = \frac{\sin x^2}{2} + C.$$

6. Calculate:

a) $\int \arcsin x\, dx$,
First by parts:

$$\int \arcsin x\, dx = x \arcsin x - \int \frac{x\, dx}{\sqrt{1-x^2}}$$

and by substitution $t = 1 - x^2$, so

$$= x \arcsin x + \frac{1}{2} \int \frac{1, dt}{\sqrt{t}} = x \arcsin x + \sqrt{t} + C = x \arcsin x + \sqrt{1-x^2} + C.$$

b) $\int \cos^3 x \sqrt{\sin x}\, dx$,
We change the variables

$$\int \cos^3 x \sqrt{\sin x}\, dx =$$

$$= \int (1 - \sin^2 x)\sqrt{\sin x} \cos x\, dx \overset{\boxed{t = \sin x,\ \frac{dt}{dx} = \cos x}}{=} \int (1-t^2)\sqrt{t} dt = \frac{2t^{\frac{3}{2}}}{3} - \frac{2t^{\frac{7}{2}}}{7} + C =$$

$$= \frac{2\sin^{\frac{3}{2}} x}{3} - \frac{2\sin^{\frac{7}{2}} x}{7} + C$$

c) $\int e^x \sin x\, dx$,
We calculate by parts twice

$$\int e^x \sin x\, dx \overset{\boxed{f(x) = e^x, g(x) = \sin x}}{=} \int f'(x) g(x)\, dx =$$

$$= e^x \sin x - \int e^x \cos x\, dx \overset{\boxed{f(x) = e^x, h(x) = \cos x}}{=}$$

$$e^x \sin x - e^x \cos x - \int e^x \sin x\, dx.$$

So

$$\int e^x \sin x\, dx = \frac{e^x}{2}(\sin x - \cos x) + C.$$

d) $\int \ln^2 x\, dx$,
Again, twice by parts

$$\int \ln^2 x\, dx \overset{\boxed{f(x) = x, g(x) = \ln^2 x}}{=} x \ln^2 x - 2 \int \frac{x}{x} \ln x\, dx \overset{\boxed{f(x) = x, h(x) = \ln x}}{=}$$

$$= x \ln^2 x - 2x \ln x + 2 \int \frac{x}{x}\, dx = x \ln^2 x - 2x \ln x + 2x + C = x(\ln x - 2\ln x + 2) + C.$$

e) $\int x\sqrt{1-x^2}\, dx$.
We change the variable

$$\int x\sqrt{1-x^2}\, dx \overset{\boxed{t = 1-x^2,\ \frac{dt}{dx} = -2x}}{=} \int \frac{-\sqrt{t} dt}{2} = -\frac{1}{2} \cdot \frac{3}{2} t^{\frac{3}{2}} + C = -\frac{\left(1-x^2\right)^{\frac{3}{2}}}{3} + C.$$