Knaster-Reichbach Theorem for 2^κ

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In the recent years the theory of the generalized Cantor and Baire spaces was extensively developed (see, e.g. [1], [2], [6], [4] and many others). An important part of the research in this subject is an attempt to transfer the results in set theory of the real line to 2^κ and κ^κ (the list of open questions can be found in [5]).

Throughout this paper, unless it is stated otherwise, we assume that κ is an uncountable regular cardinal number and $\kappa > \omega$.

We consider the space 2^κ, called **κ-Cantor space (or the generalized Cantor space)**, endowed with so called bounded topology with a base $\{[x]: x \in 2^{<\kappa}\}$, where for $x \in 2^{<\kappa}$,

$$[x] = \{f \in 2^\kappa : f \restriction \mathrm{dom} x = x\}.$$

If we additionally assume that $\kappa^{<\kappa} = \kappa$, the above base has cardinality κ. This assumption proves to be very convenient when considering the generalized Cantor space and the generalized Baire space, and is assumed throughout this note, unless stated otherwise (see e.g. [1]).

The above base consists of clopen sets. Notice also that an intersection of less than κ of basic sets is a basic set or an empty set. Therefore, an intersection of less than κ open sets is still open.

A T_1 topological space is said to be κ-additive if for any $\alpha < \kappa$, an intersection of an α-sequence of open subsets of this space is open. Various properties of κ-additive spaces were considered by R. Sikorski in [7]. Therefore, the generalized Cantor space is a zero-dimensional κ-additive space which is completely normal. The character, density and weight of 2^κ equal κ (the assumption $\kappa^{<\kappa} = \kappa$ is needed in the case of density and weight).

It is easy to see that $A \subseteq 2^\kappa$ is closed if and only if $A = [T]$ for some tree $T \subseteq 2^{<\kappa}$. Indeed, if $A = [T]$ and T is a tree, then if $x \notin A$, there exists $\alpha < \kappa$ such that $x \restriction \alpha \notin T$. Therefore $[x \restriction \alpha] \subseteq 2^\kappa \setminus A$, so A is closed. On the other hand, if A is closed, let $T = \{x \restriction \alpha : x \in A, \alpha < \kappa\}$. Then, if $a \in 2^\kappa$, and $a \restriction \alpha \in T$ for all $\alpha < \kappa$, we have that $a \in A$, since A is closed. For a closed $A \subseteq 2^\kappa$, a tree $T \subseteq 2^{<\kappa}$ such that $A = [T]$ is denoted by T_A.

A set $A \subseteq 2^\kappa$ is called **κ-closed**, if for every limit $\beta < \kappa$, and $t \in 2^\beta$ such that for all $\alpha < \beta$, $t \restriction \alpha \in T_A$, we have $t \in T_A$.

For $s, t \in 2^{\leq\kappa}$, let

$$d(s,t) = \bigcup \{\alpha < \min(\mathrm{len}(s), \mathrm{len}(t)) : \forall_{\beta<\alpha} s(\beta) = t(\beta)\}.$$

Classical **Knaster-Reichbach Theorem** (proved in [3], the authors acknowledged there that the theorem is actually due to Cz. Ryll-Nardzewski) states, that if $P, Q \subseteq 2^\omega$ are closed nowhere dense subsets of the classical Cantor space, and $h: P \to Q$ is a homeomorphism, then there exists a homeomorphism $H: 2^\omega \to 2^\omega$ such that $H \restriction P = h$.

In this note we present an analogue of Knaster-Richbach Theorem for the generalized Cantor space 2^κ. This answers an oral question of W. Kubiś.

Theorem 1 *Assume that*

(1) $P, Q \subseteq 2^\kappa$,

(2) $\langle t_\alpha \rangle_{\alpha<\kappa}, \langle s_\alpha \rangle_{\alpha<\kappa} \in (2^{<\kappa})^\kappa$,

(3) $\langle p_\alpha \rangle_{\alpha<\kappa} \in P^\kappa$, $\langle q_\alpha \rangle_{\alpha<\kappa} \in Q^\kappa$,

(4) $f, g \in \kappa^\kappa$,

(5) $h\colon P \to Q$

are such that:

(a) h *is a homeomorphism,*

(b) P *and* Q *are closed,*

(c) $2^\kappa \setminus P = \bigcup_{\alpha<\kappa}[t_\alpha]$ *and* $2^\kappa \setminus Q = \bigcup_{\alpha<\kappa}[s_\alpha]$,

(d) *for each* $\alpha < \beta < \kappa$, $[t_\alpha] \cap [t_\beta] = \varnothing$ *and* $[s_\alpha] \cap [s_\beta] = \varnothing$,

(e) f, g *are one-to-one,*

(f) *for every* $\beta < \kappa$ *there exists* $\gamma < \kappa$ *such that for all* $\gamma < \alpha < \kappa$, $d(p_\alpha, t_\alpha) > \beta$ *and* $d(q_\alpha, s_\alpha) > \beta$,

(g) *for all* $\alpha < \kappa$ *and* $p \in P$

$$d(p, t_\alpha) \leq d(p_\alpha, t_\alpha),$$

(h) *for all* $\alpha < \kappa$ *and* $q \in Q$

$$d(q, s_\alpha) \leq d(q_\alpha, s_\alpha),$$

(i) *for all* $\alpha < \kappa$

$$d(p_\alpha, t_\alpha) \leq d(h(p_\alpha), s_{f(\alpha)}),$$

(j) *for all* $\alpha < \kappa$

$$d(q_\alpha, s_\alpha) \leq d(h^{-1}(q_\alpha), t_{g(\alpha)}).$$

Then there exists a homemorphism $H\colon 2^\kappa \to 2^\kappa$ *such that*

$$H \restriction P = h.$$

Proof: First notice that there are $A_1, A_2, B_1, B_2 \subseteq \kappa$ such that

(i) $A_1 \cup A_2 = B_1 \cup B_2 = \kappa$,

(ii) $A_1 \cap A_2 = B_1 \cap B_2 = \varnothing$,

(iii) $f[A_1] = B_1$ and $g[B_2] = f[A_2]$.

For each $\alpha \in A_1$, let $f_\alpha: [t_\alpha] \to [s_{f(\alpha)}]$ be a homeomorphism. Similarly, for each $\alpha \in B_2$, let $g_\alpha: [s_\alpha] \to [t_{g(\alpha)}]$ be a homeomorphism.

Now set

$$H(x) = \begin{cases} h(x) & \text{for } x \in P, \\ f_\alpha(x) & \text{for } x \in [t_\alpha] \wedge \alpha \in A_1, \\ g_\alpha^{-1}(x) & \text{for } x \in [t_{g(\alpha)}] \wedge \alpha \in B_2. \end{cases}$$

It remains to prove that H is a homeomorphism. Actually, notice that it suffices to prove that for every $p \in P$, H is continuous at p. Let $q = h(p) = H(p) \in Q$, and let $s \in 2^{<\kappa}$ be such that $s \subseteq q$. Since h is continuous, there exists $t \in 2^{<\kappa}$ with $t \subseteq p$ such that for all $x \in P \cap [t]$, we have $h(x) \in [s]$.

Notice that

$$|\{\beta \in A_2: d(s_\beta, q_\beta) < \text{len}(t) \wedge t \subseteq t_{g(\beta)}\}| < \kappa.$$

Thus, let $\eta < \kappa$ be such that $\eta \geq \text{len}(t)$ and for all $\beta \in A_2$ such that $t_{g(\beta)} \in [p \restriction \eta]$, we have $d(s_\beta, q_\beta) \geq \text{len}(t)$.

Let $\delta = \max\{\text{len}(s), \eta\} < \kappa$. We prove that $H[[p \restriction \delta]] \subseteq [s]$. Indeed, if $x \in [p \restriction \delta] \setminus P$, then there exists $\alpha < \kappa$ such that $x \in [t_\alpha]$. We have that either $\alpha \in A_1$ or $\alpha \in A_2$.

In the first case, we get that $p_\alpha \in [p \restriction \delta]$, since $\delta \leq d(p, t_\alpha)$, but also for all $\alpha < \kappa$ and $p \in P$,

$$d(p, t_\alpha) \leq d(p_\alpha, t_\alpha).$$

Thus, $H([t_\alpha]) = [s_{f(\alpha)}] \subseteq [s]$, because for all $\alpha < \kappa$

$$d(p_\alpha, t_\alpha) \leq d(h(p_\alpha), s_{f(\alpha)}),$$

and $h(p_\alpha) \in [s]$ (and $\text{len}(s) \leq \delta$).

On the other hand, if $\alpha \in A_2$, then let $\beta \in B_2$ be such that $\alpha = g(\beta)$. Assume towards contradiction, that $s_\beta \nsubseteq s$. Then $h^{-1}(q_\beta) \nsubseteq t$, but then we get

$$d(q_\beta, s_\beta) \leq d(h^{-1}(q_\beta), t_\alpha) < \text{len}(t).$$

This is a contradiction with the choice of η, thus $s_\beta \subseteq s$.

Thus H is continuous at p. □

Lemma 1 *Assume that*

(1) $P, Q \subseteq 2^\kappa$,

(2) $\langle t_\alpha \rangle_{\alpha<\kappa}, \langle s_\alpha \rangle_{\alpha<\kappa} \in (2^{<\kappa})^\kappa$,

(3) $\langle p_\alpha \rangle_{\alpha<\kappa} \in P^\kappa$, $\langle q_\alpha \rangle_{\alpha<\kappa} \in Q^\kappa$,

(4) $h: P \to Q$

are such that:

(a) h *is a homeomorphism,*

(b) P *and* Q *are closed,*

(c) $2^\kappa \setminus P = \bigcup_{\alpha<\kappa}[t_\alpha]$ *and* $2^\kappa \setminus Q = \bigcup_{\alpha<\kappa}[s_\alpha]$,

(d) for each $\alpha < \beta < \kappa$, $[t_\alpha] \cap [t_\beta] = \varnothing$ and $[s_\alpha] \cap [s_\beta] = \varnothing$,

(e) for every $\beta < \kappa$ there exists $\gamma < \kappa$ such that for all $\gamma < \alpha < \kappa$, $d(p_\alpha, t_\alpha) > \beta$ and $d(q_\alpha, s_\alpha) > \beta$,

(f) for all $\alpha < \kappa$ and $p \in P$
$$d(p, t_\alpha) \leq d(p_\alpha, t_\alpha),$$

(g) for all $\alpha < \kappa$ and $q \in Q$
$$d(q, s_\alpha) \leq d(q_\alpha, s_\alpha),$$

(h) for all $\alpha < \kappa$
$$|\{\beta < \kappa : d(t_\alpha, p_\alpha) \leq d(s_\beta, h(p_\alpha))\}| = \kappa,$$

(i) for all $\alpha < \kappa$
$$\left|\left\{\beta < \kappa : d(s_\alpha, q_\alpha) \leq d(t_\beta, h^{-1}(q_\alpha))\right\}\right| = \kappa.$$

Then there exist $f, g \in \kappa^\kappa$ such that the premise of Theorem 1 is satisfied.

Proof: By symmetry, it is enough to prove the existence of f. We construct f by induction. For $\alpha < \kappa$, let
$$f(\alpha) = \bigcap \left(\{\beta < \kappa : d(t_\alpha, p_\alpha) \leq d(s_\beta, h(p_\alpha))\} \setminus \{f(\beta) : \beta < \alpha\}\right).$$

□

Theorem 2 *Assume that κ is strongly inaccessible. Let $P, Q \subseteq 2^\kappa$ be κ-closed nowhere dense sets, and let $h: P \to Q$ be a homeomorphism. Then there exists a homemorphism $H: 2^\kappa \to 2^\kappa$ such that*
$$H \upharpoonright P = h.$$

Proof: We start by constructing inductively sequences of sets $\langle Q_\alpha \rangle_{\alpha<\kappa}, \langle P_\alpha \rangle_{\alpha<\kappa} \subseteq (\mathcal{P}(2^{<\kappa}))^\kappa$ such that

(a) for $\alpha < \kappa$, $P_\alpha, Q_\alpha \subseteq 2^\alpha$,

(b) $\bigcup_{\alpha<\kappa} \bigcup\{[t] : t \in P_\alpha\} = 2^\kappa \setminus P$,

(c) $\bigcup_{\alpha<\kappa} \bigcup\{[t] : t \in Q_\alpha\} = 2^\kappa \setminus Q$,

(d) for all $\alpha < \kappa$, and $t \in 2^\alpha$ such that $[t] \cap P = \varnothing$ and for all $\beta < \alpha$ and all $u \in P_\beta$, $u \nsubseteq t$, $t \in P_\alpha$,

(e) for all $\alpha < \kappa$, and $s \in 2^\alpha$ such that $[s] \cap Q = \varnothing$ and for all $\beta < \alpha$ and for all $u \in Q_\beta$, $u \nsubseteq s$, $s \in Q_\alpha$.

To achieve the above for $\alpha < \kappa$, put
$$P_\alpha = \left\{t \in 2^\alpha : [t] \cap P = \varnothing \wedge \forall_{\beta<\alpha} \forall_{u \in P_\beta} u \nsubseteq t\right\},$$
and
$$Q_\alpha = \left\{t \in 2^\alpha : [t] \cap Q = \varnothing \wedge \forall_{\beta<\alpha} \forall_{u \in Q_\beta} u \nsubseteq t\right\}.$$

Notice, that since P, Q are κ-closed, for any limit ordinal $\alpha < \kappa$, $P_\alpha = Q_\alpha = \varnothing$.

Let

$$\bigcup_{\alpha<\kappa} P_\alpha = \{t_\alpha : \alpha < \kappa\},$$

and

$$\bigcup_{\alpha<\kappa} Q_\alpha = \{s_\alpha : \alpha < \kappa\}.$$

be enumerations such that for all $\alpha < \beta < \kappa$ and $\gamma, \delta < \kappa$, if $t_\gamma \in P_\alpha$ and $t_\delta \in P_\beta$, then $\gamma < \delta$, and also for all $\alpha < \beta < \kappa$ and $\gamma, \delta < \kappa$, if $s_\gamma \in Q_\alpha$ and $s_\delta \in Q_\beta$, then $\gamma < \delta$. This is possible since κ is strongly inaccessible.

Since P and Q are κ-closed, there exist $\langle p_\alpha \rangle_{\alpha<\kappa} \in P^\kappa$, $\langle q_\alpha \rangle_{\alpha<\kappa} \in Q^\kappa$ such that

(a) for all $\alpha < \kappa$ and $p \in P$

$$d(p, t_\alpha) \leq d(p_\alpha, t_\alpha),$$

(b) for all $\alpha < \kappa$ and $q \in Q$

$$d(q, s_\alpha) \leq d(q_\alpha, s_\alpha).$$

Notice also, that for all $\alpha, \gamma < \kappa$, if $t_\gamma \in P_{\alpha+1}$, $d(t_\gamma, p_\gamma) = \alpha$, and for all $\alpha, \gamma < \kappa$, if $s_\gamma \in Q_{\alpha+1}$, $d(s_\gamma, q_\gamma) = \alpha$. Thus, for every $\beta < \kappa$ there exists $\gamma < \kappa$ such that for all $\gamma < \alpha < \kappa$, $d(p_\alpha, t_\alpha) > \beta$ and $d(q_\alpha, s_\alpha) > \beta$.

Notice also, that since P is nowhere dense, we have that for every $\alpha < \kappa$

$$|\{\beta < \kappa : d(t_\alpha, p_\alpha) \leq d(s_\beta, h(p_\alpha))\}| = \kappa.$$

Indeed, if $d(t_\alpha, p_\alpha) = \gamma < \kappa$, then for every $\gamma < \delta < \kappa$, we have that there is no $\beta < \kappa$ such that $s_\beta \subseteq h(p_\alpha) \restriction \delta$, but there exists $s \in 2^{<\kappa}$ such that $h(p_\alpha) \restriction \delta \subseteq s$, and $[s] \cap Q = \varnothing$. Similarly, for every $\alpha < \kappa$,

$$\left|\left\{\beta < \kappa : d(s_\alpha, q_\alpha) \leq d(t_\beta, h^{-1}(q_\alpha))\right\}\right| = \kappa.$$

Thus, the conditions of Lemma 1 are satisfied. □

Problem 1 *Does Theorem 2 hold for uncountable regular κ which is not strongly inaccessible?*

Problem 2 *Does Theorem 2 hold for $P, Q \subseteq 2^\kappa$ which are nowhere dense and closed but not κ-closed?*

References

[1] Sy-David Friedman, Tapani Hyttinen, and Vadim Kulikov. *Generalized Descriptive Set Theory and Classification Theory*, volume 1081 of *Mem. Amer. Math. Soc.* American Mathematical Society, 2014.

[2] Sy-David Friedman and Giorgio Laguzzi. A null ideal for inaccessibles. preprint, 2014.

[3] Bronisław Knaster and Marian Reichbach. Notion d'homogénéité et prolongements des homéomorphies. *Fund. Math.*, 40:180–193, 1953.

[4] Michał Korch and Tomasz Weiss. Special subsets of the generalized Cantor space 2^κ and the generalized Baire space κ^κ. sent to Math. Logic Quart., http://duch.mimuw.edu.pl/~m_korch/wp-content/uploads/2017/09/special-kappa.pdf, 2018.

[5] Giorgio Laguzzi, Benedikt Löwe, and Ilya Sharankou. Questions on generalized Baire spaces. *Math. Logic Quart.*, 62:439–456, 2016.

[6] Saharon Shelah. A parallel to the null ideal for inaccessible λ. preprint, 2012.

[7] Roman Sikorski. Remarks on some topological spaces of high power. *Fund. Math.*, 37:125–136, 1950.