Linear algebra, WNE, 2018/2019
meetings 27. – solutions

24 January 2019

1. Consider the following system of equations

$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 = 1 \\ x_2 + 2x_2 + 3x_3 + x_4 = 3 \\ 3x_1 + 5x_2 + 8x_3 + tx_4 = 9 \\ 3x_1 + 4x_2 + tx_3 + 3x_4 = 5 \end{cases}.$$

   (a) For which real numbers $t \in \mathbb{R}$ this system is consistent?
   We transform it into echelon form

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 3 \\ 3 & 5 & 8 & t & 9 \\ 3 & 4 & t & 3 & 5 \end{array}\right] \xrightarrow{w_2 - w_1, w_2 - 3w_1, w_3 - 3w_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & t-3 & 6 \\ 0 & 1 & t-6 & 0 & 2 \end{array}\right] \xrightarrow{w_3 - 2w_2, w_4 - w_2}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & t-3 & 2 \\ 0 & 0 & t-7 & 0 & 0 \end{array}\right] \xrightarrow{w_3 \leftrightarrow w_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & t-7 & 0 & 0 \\ 0 & 0 & 0 & t-3 & 2 \end{array}\right]$$

   The inconsistency can appear in the last row, if on the left side of the equation we get zero, so if and only if $t = 3$. The system is consistent for $t \neq 3$.

   (b) For which real numbers $t \in \mathbb{R}$ this system has exactly one solution?
   The system has exactly one solution if we have a leading coefficient in every column, i.e. for $t \in \mathbb{R} \setminus \{3, 7\}$.

2. Let $V = \text{lin}((1, 1, 2, 3), (2, 3, 5, 7), (5, 6, 11, 16))$.

   (a) Find a basis and the dimension of V.
   We transform the following matrix into echelon form

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 5 & 7 \\ 5 & 6 & 11 & 16 \end{bmatrix} \xrightarrow{w_2 - 2w_1, w_3 - 5w_1} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{w_3 - w_2}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   Thus, $\dim V = 2$ and we get a basis $\{(1, 1, 2, 3), (0, 1, 1, 1)\}$.

   (b) For which real numbers $t \in \mathbb{R}$, $V = \text{lin}((1, 1, 2, 3), (2, 3, 5, 7), (5, 6, 11, 16), (1, 0, 1, t))$?
   The problem is equivalent to the question for which $t \in \mathbb{R}$, we have $(1, 0, 1, t) \in V$. So it is enough to check when the following system is consistent:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & t \end{array}\right] \xrightarrow{w_2 - w_1, w_3 - 2w_1, w_4 - 3w_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & t-3 \end{array}\right] \xrightarrow{w_3 - w_2, w_4 - w_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & t-2 \end{array}\right].$$

   Thus, $(1, 0, 1, t) \in V$ if and only if $t = 2$.

3. Let $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon 5x_1 + 2x_2 - x_3 = 0\}$ and let $\alpha_1 = (1, 0, 5), \alpha_2 = (1, 2, 9)$.

   (a) Give an example of vector α_3, such that system of vectors $\alpha_1, \alpha_2, \alpha_3$ is a basis of $\mathbb{R}^3$ and $3, 4, 1$ are the coordinates of $\beta = (9, 9, 56)$ in this basis.
   Hence, $3\alpha_1 + 4\alpha_2 + \alpha_3 = \beta$. Therefore, $\alpha_3 = \beta - 3\alpha_1 - 4\alpha_2$. Thus,

$$\alpha_3 = (9, 9, 56) - (3, 0, 15) - (4, 8, 36) = (2, 1, 5),$$

   but still we have to check whether the system $\alpha_1, \alpha_2, \alpha_3$ is linearly independent

$$\begin{bmatrix} 1 & 0 & 5 \\ 1 & 2 & 9 \\ 2 & 1 & 5 \end{bmatrix} \xrightarrow{w_2 - w_1, w_3 - 2w_1} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 4 \\ 0 & 1 & -5 \end{bmatrix} \xrightarrow{w_2 \cdot 1/2} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 1 & -5 \end{bmatrix} \xrightarrow{w_3 - w_2} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & -7 \end{bmatrix}$$

   And the answer is in the positive.

   (b) Does there exist a vector $\gamma \in V$ such that the system $\alpha_1, \alpha_2, \gamma$ is a basis of $\mathbb{R}^3$? If so, give an example of such a vector γ. If not, explain why such a vector γ does not exist.
   Let $W = \text{lin}(\alpha_1, \alpha_2)$. Notice that $\dim V = 3 - 1 = 2$. Thus, γ exists if and only if $W \subseteq V$. Hence, if and only if $\alpha_1 \notin V$ or $\alpha_2 \notin V$. Let us check: $5 - 5 = 0$, so $\alpha_1 \in V$, $5 + 4 - 9 = 0$, thus $\alpha_2 \in V$. Therefore, β does not exist.

4. Assume that the matrix of $\varphi \colon \mathbb{R}^3 \to \mathbb{R}^2$ has in bases $\mathcal{A} = \{(0, 1, 2), (0, 0, 1), (1, 1, 3)\}$ and $\mathcal{B} = \{(2, 1), (1, 0)\}$ matrix $M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 4 \\ 6 & 1 & 4 \end{bmatrix}$.

   (a) Calculate $\varphi((0, 1, 0))$.
   We have $(0, 1, 0) = (0, 1, 2) - 2(0, 0, 1)$. Thus, $\varphi((0, 1, 0)) = \varphi((0, 1, 2)) - 2\varphi((0, 0, 1)) = ((2, 1) + 6(1, 0)) - 2((2, 1) + (1, 0)) = (8, 1) - 2(3, 1) = (2, -1)$.

   (b) Find a matrix of φ in bases $\mathcal{C} = \{(1, 1, 1), (1, 2, 3), (2, 1, 1)\}$ and $\mathcal{D} = \{(0, 1), (1, 1)\}$.
   We have

$$M(\varphi)_{\mathcal{C}}^{\mathcal{D}} = M(\text{id})_{\mathcal{B}}^{\mathcal{D}} \cdot M(\varphi)_{\mathcal{A}}^{\mathcal{B}} \cdot M(\text{id})_{\mathcal{C}}^{\mathcal{A}}.$$

   And also

   - $(2, 1) = -(0, 1) + 2(1, 1)$,
   - $(1, 0) = -(0, 1) + (1, 1)$,
   - $(1, 1, 1) = 0(0, 1, 2) - 2(0, 0, 1) + (1, 1, 3)$,
   - $(1, 2, 3) = (0, 1, 2) - 2(0, 0, 1) + (1, 1, 3)$,
   - $(2, 1, 1) = -(0, 1, 2) - 3(0, 0, 1) + 2(1, 1, 3)$.

   Hence,

$$M(\text{id})_{\mathcal{B}}^{\mathcal{D}} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

   and

$$M(\text{id})_{\mathcal{C}}^{\mathcal{A}} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & -3 \\ 1 & 1 & 2 \end{bmatrix}.$$

   Thus,

$$M(\varphi)_{\mathcal{C}}^{\mathcal{D}} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 4 \\ 6 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & -3 \\ 1 & 1 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} -7 & -2 & -8 \\ 8 & 3 & 12 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & -3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -11 & -3 \\ 6 & 14 & 7 \end{bmatrix}.$$

5. Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$, and let $A^{150} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$.

(a) Check whether matrix A is diagonalizable. If so, find diagonal matrix similar to A.
The characteristic polynomial of A is:

$$w(\lambda) = (1-\lambda)(4-\lambda) + 2 = (\lambda-2)(\lambda-3).$$

Hence, we get 2 eigenvalues in 2-dimensional space, so there definitely is a basis which consists of eigenvectors, in which the matrix takes the following form

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

(b) Calculate x.
Let us find this basis of eigenvector. For $\lambda = 2$, we get $-x - 2y = 0$, hence eigenvector $(-2, 1)$. For $\lambda = 3$, $x + y = 0$, so $(-1, 1)$. So $\mathcal{A} = \{(-2, 1), (-1, 1)\}$ is a basis consisting of eigenvectors, and

$$(1, 0) = -(-2, 1) + (-1, 1),$$

$$(0, 1) = -(-2, 1) + 2(-1, 1),$$

so

$$A^{150} = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2^{150} & 0 \\ 0 & 3^{150} \end{bmatrix} \cdot \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2^{151} & -3^{150} \\ 2^{150} & 3^{150} \end{bmatrix} \cdot \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}.$$

Thus, $x = 2^{151} - 2 \cdot 3^{150}$.

6. Consider hyperplane $H \subseteq \mathbb{R}^4$, $H = (1, 2, 1, 1) + \text{lin}((1, 1, 0, 2), (1, 2, 0, 3), (1, 1, 1, 4))$ and a line $L \subseteq \mathbb{R}^4$ going though $(1, 0, 1, 0)$ and $(3, 1, 2, 4)$.

(a) Find and equation describing H.
First, we calculate an equation for $T(H) = \text{lin}((1, 1, 0, 2), (1, 2, 0, 3), (1, 1, 1, 4))$, so we solve a system of equations to find a basis of $T(H)^\perp$:

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 3 \\ 1 & 1 & 1 & 4 \end{bmatrix} \xrightarrow{w_2 - w_1, w_3 - w_1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{w_1 - w_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

So we get a basis of $T(H)^\perp$: $\{(-1, -1, -4, 1)\}$, thus we get the following equation for $T(H)$:

$$-x - y - 4z + w = 0.$$

Thus,

$$-x - y - 4z + w = -6.$$

is an equation for H.

(b) Find a parametrization of L and the point of intersection of L and H.
We have $L = (1, 0, 1, 0) + \text{lin}((3, 1, 2, 4) - (1, 0, 1, 0)) = (1, 0, 1, 0) + \text{lin}((2, 1, 1, 4)) = \{(1 + 2t, t, 1 + t, 4t) : t \in \mathbb{R}\}$.
So the point of intersection of L and H is a point $(1+2t, t, 1+t, 4t)$ for such value of t, that it satisfies $-x - y - 4z + w = -6$. Hence, $-1 - 2t - t - 4 - 4t + 4t = -6$, so $-5 - 3t = -6$, and we get $t = 1/3$, and so $(5/3, 1/3, 4/3, 4/3)$ is the point of intersection.

7. Consider the following linear programming problem $4x_1 + x_2 + 2x_3 + x_4 + 5x_5 \to \min$ with constraints:

$$\begin{cases} 2x_1 + x_2 + x_3 + x_5 = 2 \\ 3x_1 + x_2 + 3x_3 + x_4 + 4x_5 = 7 \\ x_1, x_2, x_3, x_4, x_5 \geqslant 0 \end{cases}.$$

(a) Find whether $\{2, 4\}$ is a feasible set of basic variables.
We get $x_2 = 2$ and $x_2 + x_4 = 7$, thus $x_4 = 5$, so the basic solution is $(0, 2, 0, 5, 0)$, and it is feasible.

(b) Solve this problem using simplex method.

Since we know a feasible basic solution we start with it

$$\left[\begin{array}{ccccc|c} 4 & 1 & 2 & 1 & 5 & 0 \\ \hline 2 & 1 & 1 & 0 & 1 & 2 \\ 3 & 1 & 3 & 1 & 4 & 7 \end{array}\right] \xrightarrow{w_0 - w_1, w_2 - w_1} \left[\begin{array}{ccccc|c} 2 & 0 & 1 & 1 & 4 & -2 \\ \hline 2 & 1 & 1 & 0 & 1 & 2 \\ 1 & 0 & 2 & 1 & 3 & 5 \end{array}\right] \xrightarrow{w_0 - w_2}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 1 & -7 \\ \hline 2 & 1 & 1 & 0 & 1 & 2 \\ 1 & 0 & 2 & 1 & 3 & 5 \end{array}\right]$$

Hence we are in $(0, 2, 0, 5, 0)$ (basic variables: x_2, x_4, cost 7). We have only one improving edge. Thus, x_3 will become basic. The bounds are $2/1$ and $5/2$. The first is lower, so the first row is chosen, and x_2 is dropped from the set of basic variables.

$$\xrightarrow{w_0 + w_1, w_2 - 2w_1} \left[\begin{array}{ccccc|c} 3 & 1 & 0 & 0 & 2 & -5 \\ \hline 2 & 1 & 1 & 0 & 1 & 2 \\ -3 & -2 & 0 & 1 & 1 & 1 \end{array}\right]$$

We are in $(0, 0, 2, 1, 0)$ (basic variables x_3, x_4, cost: 5). There are no improving edges so the vertex is optimal.

8. Consider quadratic forms $q_1 \colon \mathbb{R}^3 \to \mathbb{R}$, $q_1(x_1, x_2, x_3) = x_1^2 + 5x_2^2 + 7x_3^2 + 4x_1x_2 - 2x_1x_3$ and $q_2 \colon \mathbb{R}^3 \to \mathbb{R}$, $q_2(x_1, x_2, x_3) = 2x_1x_2 + 2x_2x_3$.

(a) Check whether q_1 is positively definite?

We have $A = M(q_1) = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & 7 \end{bmatrix}$. The subsequent determinants are

$$\det A_1 = 1 > 0,$$

$$\det A_2 = 5 - 4 = 1 > 0,$$

$$\det A_3 = 35 - 5 - 28 = 2 > 0,$$

so from the Silverster's Criterion we have that this form is positively definite.

(b) Check whether q_2 is negatively semidefinite?

We have $M(q_2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. The characteristic polynomial $w(\lambda) = -\lambda^3 + \lambda + \lambda = -\lambda(\lambda^2 - 2)$ gives eigenvalues $0, \sqrt{2}, -\sqrt{2}$, so the form is not negatively semidefinite since $\sqrt{2} > 0$.