Linear algebra, WNE, 2018/2019
meeting 1. – solutions to supplementary problems

2 October 2018

1. Describe the set of solutions depending on parameter $a \in \mathbb{R}$:

$$\begin{cases} x + 2y + (a+3)z = 8 \\ 2x + 3y + (a+4)z = 12 \\ 3x + (6a+5)y + 7z = 20 \end{cases}$$

The first equation implies that $x = 8 - 2y - az - 3z$. Therefore,

$$\begin{cases} 16 - 4y - 2az - 6z + 3y + az + 4z = 12 \\ 24 - 6y - 3az - 9z + 6ay + 5y + 7z = 20 \end{cases},$$

so:

$$\begin{cases} -y - 2z - az = -4 \\ -y + 6ay - 2z - 3az = -4 \end{cases},$$

thus, $y = 4 - 2z - az$, and therefore $-4 + 2z + az + 24a - 12az - 6a^2z - 2a - 3az = -4$, which means that: $-6a^2z - 14az = -24a$, and $3a^2z + 7az = 12a$.

If $a = 0$, then this equation is valid regardless the value of z and the first equation is just a sum of the the other two. Therefore for any z, we get $y = 4 - 2z$ and $x = 8 - 2y - 3z = z$. Thus, in this case we get infinitely many solutions and each of them is of form $(z, 4 - 2z, z)$ for any $z \in \mathbb{R}$.

But if $a \neq 0$, we can divide $z(3a^2 + 7a) = 12a$ by a, and get $z(3a + 7) = 12$. In this case, if $3a + 7 = 0$ (i.e. if $a = \frac{-7}{3}$), we get a contradiction and the set of equations does not have any solutions.

If on the other hand $a \neq 0$ and $a \neq \frac{-7}{3}$, then finally $z = \frac{12}{3a+7}$. Thus, $y = \frac{12a+28-24-12a}{3a+7} = \frac{4}{3a+7}$ and $x = \frac{24a+56-8-12a-36}{3a+7} = \frac{12a+12}{3a+7}$. In this case we get only one solution.

2. Find all roots of $x^4 + 3x^3 - 12x^2 - 20x + 48$. Draw a rough graph of this polynomial.

One can easily see that 2 is a root. Dividing by $(x - 2)$, we get $(x^3 + 5x^2 - 2x - 24)(x - 2)$. 2 is again a root, and we get $(x^2 + 7x + 12)(x - 2)^2$. Thus, $\Delta = 49 - 48 = 1$, and the other roots are -4 and -3. The graph looks like this: