Linear algebra, WNE, 2018/2019
meeting 26. – solutions

22 January 2019

1. Check whether the following quadratic forms are positive definite or negative definite

   - $q\colon \mathbb{R}^2 \to \mathbb{R}, q((x_1, x_2)) = -x_1^2 + 4x_1x_2 - 5x_2^2$,
   - $q\colon \mathbb{R}^2 \to \mathbb{R}, q((x, y, z)) = x^2 + 2y^2 + 2z^2 + 2xy + 2xz$.
   - $q\colon \mathbb{R}^4 \to \mathbb{R}, q((a, b, c, d)) = a^2 + 3b^2 + 5c^2 + 7d^2 + 2ab + 2ac + 2ad + 2bc + 6bd + 4cd$.

   - Matrix of this form is $A = \begin{bmatrix} -1 & 2 \\ 2 & -5 \end{bmatrix}$. We check the determinants of leading principal minors $\det A_1 = -1 < 0$, $\det A_2 = 5 - 4 = 1 > 0$, so by Silvester's Criterion, the form is not positively definite, but it is negatively definite.
   - Matrix of this form is $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$. We check the determinants of leading principal minors $\det A_1 = 1 > 0, \det A_2 = 2 - 1 = 1 > 0, \det A_3 = 4 + 0 + 0 - 2 - 2 = 0$, so by Silvester's Criterion, the form is neither positively nor negatively definite.
   - Matrix of this form is $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 3 \\ 1 & 1 & 5 & 2 \\ 1 & 3 & 2 & 7 \end{bmatrix}$. We check the determinants of leading principal minors $\det A_1 = 1 > 0, \det A_2 = 3 - 1 = 2 > 0, \det A_3 = 15 + 1 + 1 - 3 - 1 - 5 = 8 > 0 \det A_4 = 30 > 0$, so by Silvester's Criterion, the form is positively definite.

2. For which real numbers $r \in \mathbb{R}$ the quadratic form $q\colon \mathbb{R}^3 \to \mathbb{R}, q((x, y, z)) = -x^2 + ry^2 + rz^2 + 4xy + 2yz$ is negatively definite?

   Matrix of this form is $A = \begin{bmatrix} -1 & 2 & 0 \\ 2 & r & 1 \\ 0 & 1 & r \end{bmatrix}$. We check the determinants of leading principal minors $\det A_1 = -1, \det A_2 = -r-4, \det A_3 = -r^2+1-4r$. By Silvester's criterion we need that $-r-4 > 0$ i $r^2+4r-1 > 0$. The first condition implies $r < -4$, and the second $r < -2 - \sqrt{5}$ or $r > -2 + \sqrt{5}$. Hence, finally q is negatively definite for $r < -2 - \sqrt{5}$.

3. Using eigenvalues check whether the following quadratic forms are positively or negatively definite, positively or negatively semidefinite, or indefinite.

   - $q\colon \mathbb{R}^2 \to \mathbb{R}, q((x, y)) = x^2 + 9y^2 + 6xy$,
   - $q\colon \mathbb{R}^4 \to \mathbb{R}, q((x, y, z, t)) = 5x^2 + 5y^2 + 4z^2 + t^2 + 6xy + 4zt$.

   - Matrix of this form is $A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$, so the deterministic polynomial is $w(\lambda) = (1 - \lambda)(9 - \lambda) - 9 = \lambda^2 - 10\lambda = \lambda(\lambda - 10)$. We get eigenvalues 0 and 10. Hence this form is neither positively, nor negatively definite. It is positively semidefinite, but not negatively semidefinite. It is not indefinite.
   - Matrix of this form is $A = \begin{bmatrix} 5 & 3 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, so the deterministic polynomial is $w(\lambda) = ((5 - \lambda)^2 - 9)((4 - \lambda)(1 - \lambda) - 4) = (\lambda^2 - 10\lambda + 16)(\lambda^2 - 5\lambda) = \lambda(\lambda - 2)(\lambda - 5)(\lambda - 8)$. We get eigenvalues

$0, 2, 5, 8$, Hence this form is neither positively, nor negatively definite. It is positively semidefinite, but not negatively semidefinite. It is not indefinite.

4. For which real numbers $r, s \in \mathbb{R}$ the quadratic form $q\colon \mathbb{R}^3 \to \mathbb{R}, q((x, y, z)) = x^2 + 2rxy + 4y^2 + sz^2$ is:

   - positively definite?
   - positively semidefinite?
   - negatively definite?
   - negatively semidefinite?
   - indefinite?

   Matrix of this form is $A = \begin{bmatrix} 1 & r & 0 \\ r & 4 & 0 \\ 0 & 0 & s \end{bmatrix}$.
   Characteristic polynomial $(s - \lambda)((1 - \lambda)(4 - \lambda) - r^2)$. We get eigenvalues

   $$s, \frac{5 - \sqrt{9 + 4r^2}}{2}, \frac{5 + \sqrt{9 + 4r^2}}{2},$$

   out of which the last value is always positive, and the second is > 0 for $r \in (-2, 2)$, $= 0$ for $r = \pm 2$ and < 0 otherwise. Hence, the form is

   - positively definite if $s > 0$ and $r \in (-2, 2)$,
   - positively semidefinite if $s \geqslant 0$ and $r \in [-2, 2]$,
   - negatively definite – never,
   - negatively semidefinite – never,
   - indefinite if $s < 0$ or $r \notin (-2, 2)$.