Linear algebra , WNE, 2018/2019
meeting 25. – solutions

17 January 2019

1. Using simplex method solve the following linear programming problems

   - $2x_1 - x_2 \to max$, with constraints:
     $4x_1 + 4x_2 \leqslant 12$,
     $x_1 \leqslant 2, x_2 \leqslant 2$,
     $x_1 \geqslant 0$, $x_2 \geqslant 0$.
     First, we write out this problem in a standard form $-2x_1 + x_2 \to min$, with constraints:
     $4x_1 + 4x_2 + x_3 = 12$,
     $x_1 + x_4 = 2$,
     $x_2 + x_5 = 2$,
     $x_1, x_2, x_3, x_4, x_5 \geqslant 0$.
     Simplex array for basic variables x_3, x_4, x_5:

$$\begin{array}{ccccc|c} -2 & 1 & 0 & 0 & 0 & 0 \\ \hline 4 & 4 & 1 & 0 & 0 & 12 \\ 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{array}$$

     Vertex $(0, 0, 12, 2, 2)$ has one improving edge and one aggravating edge. The bounds for the improving edge are $\mathcal{F} = \{3, 2\}$, where the lower value is related to the second row. So we drop x_4 from the basic variables and add x_1:

$$\xrightarrow{w_0 + 2w_2, w_1 - 4w_2} \begin{array}{ccccc|c} 0 & 1 & 0 & 2 & 0 & 4 \\ \hline 0 & 4 & 1 & -4 & 0 & 4 \\ 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{array}$$

     Vertex $(2, 0, 4, 0, 2)$ has only aggravating edges, so we are in an optimal vertex. The cost value is -4, so in the original problem we get $(2, 0)$ with objective value 4.

   - $3x - 2y \to max$, with constraints
     $-3x + 2y \geqslant 8$,
     $x - y \leqslant 0$,
     $x \geqslant 0, y \geqslant 0$.
     First, we write out the problem in a standard form $-3x + 2y \to \min$
     $-3x + 2y - a = 8$
     $x - y + b = 0$
     $x, y, a, b \geqslant 0$.
     We do not see any feasible basic solution so we add an additional variable m and we modify the problem: $-3x + 2y + Mm \to \min$
     $-3x + 2y - a + m = 8$
     $x - y + b = 0$
     $x, y, a, b, m \geqslant 0$.
     where M should be considered to be a very big number so the algorithm should drop the variable m, to minimize the cost value. So we get the following simplex array

$$\left[\begin{array}{ccccc|c} -3 & 2 & 0 & 0 & M & 0 \\ \hline -3 & 2 & -1 & 0 & 1 & 8 \\ 1 & -1 & 0 & 1 & 0 & 0 \end{array}\right]$$

and after the reduction of columns related to basic variables (b and m) we get:

$$\left[\begin{array}{ccccc|c} -3+3M & 2-2M & M & 0 & 0 & -8M \\ \hline -3 & 2 & -1 & 0 & 1 & 8 \\ 1 & -1 & 0 & 1 & 0 & 0 \end{array}\right]$$

We have a negative value in the second column, so we we add y to basic variables. Since in the third row we have a negative value, we can choose only the second row, and indeed it means that we are dropping the artificial variable m from the set of basic variables. After reduction ($w_2 \cdot \frac{1}{2}, w_1+(2+2M)w_2, w_3+w_2$ we get:

$$\left[\begin{array}{ccccc|c} 0 & 0 & 1 & 0 & -1+M & -8 \\ \hline \frac{-3}{2} & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 4 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & \frac{1}{2} & 4 \end{array}\right]$$

We do not have any improving edges, so we are in an optimal vertex $x = 0, y = 4$, and the cost value is 8, so the objective in the original problem is -8.

- $8x + u \to max$, with constraints:
  $2x + 4y + 8u = 10$,
  $3y + z - u = 3$,
  $t + 6u = 12$,
  $x, y, z, t, u \geqslant 0$.
  First we write out the problem in a standard form $-8x - u \to \min$, with constraints:
  $2x + 4y + 8u = 10$,
  $3y + z - u = 3$,
  $t + 6u = 12$,
  $x, y, z, t, u \geqslant 0$.
  We can take basic variables x, z, t and we get the following simplex array then

$$\begin{array}{ccccc|c} -8 & 0 & 0 & 0 & -1 & 0 \\ \hline 1 & 2 & 0 & 0 & 4 & 5 \\ 0 & 3 & 1 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 & 6 & 12 \end{array} \xrightarrow{w_0 + 8w_1 [} \begin{array}{ccccc|c} 0 & 16 & 0 & 0 & 31 & 40 \\ \hline 1 & 2 & 0 & 0 & 4 & 5 \\ 0 & 3 & 1 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 & 6 & 12 \end{array}$$

The vertex $(5, 0, 3, 12, 0)$ has only aggravating edges, so we are in an optimal vertex, with cost -40, so in the original problem the objective value is 40.