Linear algebra, WNE, 2018/2019
meeting 24. – homework solutions

10 January 2019

Group 8:00

1. Find a standard form of the linear programming problem $5a + 6b - 3c \to max$ with constraints
   $a + 2b - c = 19$,
   $a + c \geqslant 0$,
   $b \geqslant 0$, $c \geqslant 0$.

   Solution:
   $-5a^+ + 5a^- - 6b + 3c \to \min$
   $a^+ - a^- + 2b - c = 19$,
   $a^+ - a^- + c - d = 0$,
   $a^+, a^-, b, c, d \geqslant 0$.

2. For the following system of equations find all basic solutions

$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 = 2 \\ 2x_1 + x_2 + 4x_3 + x_4 = 1 \\ 3x_1 - x_2 + 6x_3 - x_4 = 0 \end{cases}$$

   The system is inconsistent, so there are no basic solutions.

3. Solve the following linear programming problem using simplex method
   $-x_1 - 2x_2 \to min$, with constraints:
   $4x_1 + 4x_2 \leqslant 12$,
   $x_1 \leqslant 2, x_2 \leqslant 2$,
   $x_1 \geqslant 0$, $x_2 \geqslant 0$.

   Solution: the standard form is $-x_1 - 2x_2 \to \min$, with constraints: $4x_1 + 4x_2 + x_3 = 12$,
   $x_1 + x_4 = 2$,
   $x_2 + x_5 = 2$,
   $x_1, x_2, x_3, x_4, x_5 \geqslant 0$.

   We write out the simplex array (the last three variables are basic):

$$\begin{array}{ccccc|c} -1 & -2 & 0 & 0 & 0 & 0 \\ \hline 4 & 4 & 1 & 0 & 0 & 12 \\ 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{array}$$

   Vertex $(0, 0, 12, 2, 2)$. We get two improving edges and choose the one from the second column. Its bounds are $\mathcal{F} = \{3, 2\}$, So it of length 2 and x_5 ceases to be basis in favour of x_2:

$$\xrightarrow{w_0 + 2w_3, w_1 - 4w_3} \begin{array}{ccccc|c} -1 & 0 & 0 & 0 & 2 & 4 \\ \hline 4 & 0 & 1 & 0 & -4 & 4 \\ 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{array}$$

Vertex $(0,2,4,2,0)$. We have only one improving edge, related to the second column. Its bounds are $\mathcal{F} = \{1,2\}$, so 1 and x_4 ceases to be basic, and x_1 becomes a basic variable

$$\xrightarrow{w_1 \cdot \frac{1}{4}} \begin{array}{ccccc|c} -1 & 0 & 0 & 0 & 2 & 4 \\ \hline 1 & 0 & \frac{1}{4} & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{array} \xrightarrow{w_0 + w_1, w_2 - w_1} \begin{array}{ccccc|c} 0 & 0 & \frac{1}{4} & 0 & 1 & 5 \\ \hline 1 & 0 & \frac{1}{4} & 0 & -1 & 1 \\ 0 & 0 & -\frac{1}{4} & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{array}$$

Vertex $(1,2,1,0,0)$ with cost value -5 is optimal because we have only aggravating edges.
It translates to solution $(1,2)$ in the original problem.

Group 9:45

1. Find a standard form $5a - 6b - 3c \to max$ with constraints
$a + 2b - c = 19$,
$a + c \leqslant 0$,
$b \geqslant 0$, $c \geqslant 0$.

   Solution:
$-5a^+ + 5a^- + 6b - 3c \to \min$
$a^+ - a^- + 2b - c = 19$,
$a^+ - a^- + c + d = 0$,
$a^+, a^-, b, c, d \geqslant 0$.

2. Find all the basic solutions of the following system of equations

$$\begin{cases} x_1 + 2x_2 + 3x_3 + x_4 = 0 \\ 2x_1 + 5x_2 + x_3 + x_4 = 1 \\ x_1 + x_2 + x_3 - x_4 = 2 \end{cases}$$

   Solution:
In the echelon form we get

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & -5 & -1 & 1 \\ 0 & 0 & -7 & -3 & 3 \end{array}\right]$$

   Next, we calculate basic solutions

   - basic variables x_1, x_2, x_3, non-basic $x_4 = 0$, we get $(\frac{5}{7}, -\frac{1}{7}, -\frac{3}{7}, 0)$ – infeasible basic solution,
   - basic variables x_1, x_2, x_4, non-basic $x_3 = 0$, we get $(1,0,0,-1)$ – infeasible basic solution,
   - basic variables x_1, x_3, x_4, non-basic $x_2 = 0$, we get $(1,0,0,-1)$ – infeasible basic solution,
   - basic variables x_2, x_3, x_4, non-basic $x_1 = 0$, we get $(0, \frac{4}{9}, \frac{1}{6}, -\frac{25}{18})$ – infeasible basic solution,

   So we have 3 basic solutions all of which are infeasible $(\frac{5}{7}, -\frac{1}{7}, -\frac{3}{7}, 0), (1,0,0,-1), (0, \frac{4}{9}, \frac{1}{6}, -\frac{25}{18})$.

3. Solve the following linear programming problem using the simplex method:
$2x_1 + 2x_2 \to max$, with constraints:
$8x_1 + 4x_2 \leqslant 12$,
$2x_1 \leqslant 2, x_2 \leqslant 2$,
$x_1 \geqslant 0$, $x_2 \geqslant 0$.

   Solution:
the standard form: $-2x_1 - 2x_2 \to \min$, with constraints: $8x_1 + 4x_2 + x_3 = 12$,
$2x_1 + x_4 = 2$,
$x_2 + x_5 = 2$,
$x_1, x_2, x_3, x_4, x_5 \geqslant 0$.

   Simplex array (the three last variables are basic):

$$\begin{array}{ccccc|c} -2 & -2 & 0 & 0 & 0 & 0 \\ \hline 8 & 4 & 1 & 0 & 0 & 12 \\ 2 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{array}$$

Vertex $(0,0,12,2,2)$. Thus we have two improving edges, and we choose the one from the second column. The bound are $\mathcal{F}=\{3,2\}$, so it has length 2 and x_5 ceases to be basic, x_2 is a new basic variable:

$$\xrightarrow{w_0+2w_3,\, w_1-4w_3} \begin{array}{ccccc|c} -2 & 0 & 0 & 0 & 2 & 4 \\ \hline 8 & 0 & 1 & 0 & -4 & 4 \\ 2 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{array}$$

Vertex $(0,2,4,2,0)$. We have only one improving edge, the one in the first column. The bounds are $\mathcal{F}=\{1,2\}$, so 1 and x_4 is dropped from the basic variables set. x_1 is basic now:

$$\xrightarrow{w_1\cdot\frac{1}{8}} \begin{array}{ccccc|c} -2 & 0 & 0 & 0 & 2 & 4 \\ \hline 1 & 0 & \frac{1}{8} & 0 & -\frac{1}{2} & \frac{1}{2} \\ 2 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{array} \xrightarrow{w_0+2w_1,\, w_2-2w_1} \begin{array}{ccccc|c} 0 & 0 & \frac{1}{4} & 0 & 1 & 5 \\ \hline 1 & 0 & \frac{1}{8} & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{4} & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{array}$$

Vertex $(\frac{1}{2},2,1,0,0)$ with cost value -5, is optimal because all the edges are aggravating This vertex translates to solution $(\frac{1}{2},2)$ of the original problem.