Linear algebra, WNE, 2017/2018
before the 3rd test

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Spis treści

1 Finding eigenvalues and bases of eigenspaces

1.1 Method

If you have tried to imagine a linear map of a plane, you usually imagine that it stretches or squeezes the plane along some directions. It would be nice to know whether a given linear map $\varphi\colon V \to V$ (such a map, which the same domain and range, is called an endomorphism), it is actually of this type. In other words we would like to know whether there exists a non-zero vector v and a scalar λ, such that φ simply multiples v by λ (so it stretches or squeezes the space in the direction of v), so:

$$\varphi(v) = \lambda v.$$

If v and λ have such properties, then v is said to be an eigenvector of φ and λ an eigenvalue.

Notice that if λ is an eigenvalue of a map φ and v is its eigenvector, then $\varphi(v) - \lambda v = 0$. Therefore if M is a matrix of φ (in standard basis), then

$$0 = Mv - \lambda v = Mv - \lambda Iv = (M - \lambda I)v$$

, where I is the identity matrix.

Since multiplication of a matrix by a vector gives a linear combination of its columns and v is a non-zero vector, we see that the columns of $M - \lambda I$ can be non-trivially combined to get the zero vector! It is possible if and only if $\det(M - \lambda I) = 0$. Therefore, $\det(M = \lambda I)$ is called the characteristic polynomial of φ.

How to find the eigenvalues of a map? Simply one needs to solve the equation $\det(M - \lambda I) = 0$.

Now let's find eigenvectors related to subsequent eigenvalues. Notice that since φ is a linear map, if v, v' are eigenvectors for an eigenvalue λ, then for any scalar a also av and $v + v'$ are eigenvectors for λ. Therefore, the set of all eigenvectors for λ forms a linear subspace. Notice that v satisfies the equation

$$(M - \lambda I)v = 0$$

so the space of eigenvectors (i.e. eigenspace) for λ (denoted as $V_{(\lambda)}$) is given by the following system of equations:

$$(M - \lambda I)v = 0$$

and we can easily find its basis.

1.2 Example

E.g let $\varphi(x, y, z) = (2x, x + y, -x + z)$. Then:

$$M(\varphi)_{st}^{st} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Therefore:

$$M(\varphi)_{st}^{st} - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{bmatrix}.$$

So we have to solve the following:

$$\det(M(\varphi)_{st}^{st} - \lambda I) = (2-\lambda)(1-\lambda)^2 = 0$$

And therefore the eigenvalues are: 2 and 1.

et us find a basis of $V_{(1)}$, so let $\lambda = 1$. Then:

$$M(\varphi)_{st}^{st} - \lambda I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Therefore, we have the following system of equations:

$$\begin{cases} x = 0 \\ x = 0 \\ -x = 0 \end{cases}$$

The space of solutions is $V_{(1)} = \{(0, y, z) \colon y, z \in \mathbb{R}\}$, and its basis is $(0,1,0), (0,0,1)$. Indeed, $\varphi((0,1,0)) = 1 \cdot (0,1,0)$ and $\varphi((0,0,1)) = 1 \cdot (0,0,1)$.

Let's find a basis of $V_{(2)}$, so let $\lambda = 2$. Then:

$$M(\varphi)_{st}^{st} - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

The system of equations:

$$\begin{cases} 0 = 0 \\ x - y = 0 \\ -x - z = 0 \end{cases}$$

In the reduced „stair-like" form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The space of solutions is $V_{(2)} = \{(-z, -z, z) \colon y, z \in \mathbb{R}\}$, and its basis is $(-1,-1,1)$. Indeed, $\varphi((-1,-1,1)) = (-2,-2,2) = 2 \cdot (-1,-1,1)$.

1.3 Exemplary problems

Find eigenvalues on bases of eigenspaces of the following linear transformations:

- $\varphi \colon \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi((x,y,z)) = (x + 2y, 3x + 4y, 5z)$,
- $\psi \colon \mathbb{R}^2 \to \mathbb{R}^2$, $\psi((a,b)) = (3a + b, 5b)$,
- $\phi \colon \mathbb{R}^4 \to \mathbb{R}^4$, $\phi((x,y,z,t)) = (-y, x, 2z - t, -z + 2t)$.

2 Finding a basis consisting of eigenvectors of a linear transformation and its matrix in this basis

2.1 Method

If the sum of dimensions of spaces related to the eigenvalues of a given map equals the dimension of the whole space (as in our example: $1+2=3$), then the basis of the whole space which consists of the vectors from the bases of subspaces related to the eigenvalues is called an eigenvector basis.

If a map has an eigenvector basis, then it can be actually described by means of squeezing and stretching in the directions of eigenvectors. Notice that the matrix of such a map in an eigenvector basis is an diagonal matrix (has non-zero elements only on its diagonal) with eigenvalues related to subsequent eigenvectors on its diagonal.

It may happen that a map has no eigenvectors (e.g. a rotation of the plane) or that the subspaces of eigenvectors are to small (e.g. a 10 degree rotation of a three-dimensional space around an axis had only one-dimensional space of eigenvectors).

2.2 Example

In the example from the previous topic, putting the vectors from bases of eigenspaces we get

$$\mathcal{A} = \{(0,1,0),(0,0,1),(-1,-1,1)\},$$

which is a basis of the whole space.

Then,

$$M(\varphi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

On the other hand, $\varphi(x,y) = (x-y, x+3y)$, so $(1-\lambda)(3-\lambda)+1 = \lambda^2 - 4\lambda + 4$ is the characteristic polynomial, so the only eigenvalue is $\lambda = 2$. $V_{(2)}$ is spanned by $\{(1,-1)\}$, so there is only one direction, and there is no basis of the whole space, consisting of eigenvectors.

2.3 Exemplary problems

Check whether there exists a basis of the whole space consisting of eigenvectors. If so, find a matrix of the transformation in this basis.

- $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi((x,y,z)) = (x+2y, 3x+4y, 5z)$,
- $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\psi((a,b)) = (3a+b, 5b)$,
- $\phi\colon \mathbb{R}^4 \to \mathbb{R}^4$, $\phi((x,y,z,t)) = (-y, x, 2z-t, -z+2t)$.

3 Diagonalization of a matrix

3.1 Method

A matrix M is diagonalizable, if there exists a matrix C, such that:

$$M = C \cdot D \cdot C^{-1},$$

where D is a diagonal matrix.

How to check it and diagonalize a matrix if it is possible? Simply consider a linear map φ such that M is its matrix in standard basis. Matrix M is diagonalizable, if and only if φ has eigenvector basis $\mathcal{A}$. Then:

$$M = M(id)_{\mathcal{A}}^{st} \cdot D \cdot (M(id)_{\mathcal{A}}^{st})^{-1}$$

and $D = M(\varphi)_{\mathcal{A}}^{\mathcal{A}}$, since $(M(id)_{\mathcal{A}}^{st})^{-1} = M(id)_{st}^{\mathcal{A}}$.

3.2 Example

From the example considered in the previous topic, we know that we know that matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

is diagonalizable, since φ related to this matrix has an eigenvector basis. furthermore in this case:

$$D = M(\varphi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and

$$C = M(id)_{\mathcal{A}}^{st} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

3.3 Exemplary problems

Check whether the following matrices M are diagonalizable. If so, find matrices C and D, such that D is diagonal, and $M = C \cdot D \cdot C^{-1}$.

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

4 Calculating powers of diagonalizable matrices

4.1 Method

Diagonalization of a matrix has the following interesting matrix. We can use it to calculate a power of a matrix, if it is diagonalizable. Notice that if $\mathcal{A}$ is a basis, then:

$$\left(M(\varphi)_{\mathcal{A}}^{\mathcal{A}}\right)^n = M\left(\underbrace{\varphi \circ \ldots \circ \varphi}_{n}\right)_{\mathcal{A}}^{\mathcal{A}},$$

Therefore:

$$(M(\varphi)_{st}^{st})^n = M\left(\underbrace{\varphi \circ \ldots \circ \varphi}_{n}\right)_{st}^{st} =$$

$$= M(id)_{\mathcal{A}}^{st} \cdot M\left(\underbrace{\varphi \circ \ldots \circ \varphi}_{n}\right)_{\mathcal{A}}^{\mathcal{A}} \cdot M(id)_{st}^{\mathcal{A}} = M(id)_{\mathcal{A}}^{st} \cdot \left(M\left(\varphi\right)_{\mathcal{A}}^{\mathcal{A}}\right)^n \cdot M(id)_{st}^{\mathcal{A}}$$

but if $\mathcal{A}$ is a basis of eigenvectors, then $M\left(\varphi\right)_{\mathcal{A}}^{\mathcal{A}}$ is a diagonal matrix so calculating its power is simply calculating powers of the elements on the diagonal.

4.2 Example

Let us use again the same example as before. Let us calculate:

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^5$$

We have:

$$C = M(id)_{\mathcal{A}}^{st} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Therefore:

$$C^{-1} = M(id)_{st}^{\mathcal{A}} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

So:

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^5 = C \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^5 \cdot C^{-1} = C \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{bmatrix}^5 \cdot C^{-1} = \begin{bmatrix} 32 & 0 & 0 \\ 31 & 1 & 0 \\ -31 & 0 & 1 \end{bmatrix}.$$

4.3 Exemplary problems

Calculate

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}^{2017}.$$

5 Calculating length of a vector

5.1 Method

To study angles between vectors it will be convenient to use scalar product. The scalar product of two vectors α, β, is $\langle \alpha, \beta \rangle$. Standard scalar product in $\mathbb{R}^n$ is sum of products on subsequent places, so e.g.:

$$\langle (1, 2, -1), (2, 0, 1) \rangle = 1 \cdot 2 + 2 \cdot 0 + (-1) \cdot 1 = 2 + 0 - 1 = 1.$$

By Pitagoras Theorem it is easy to see that $\langle \alpha, \alpha \rangle$ is the square of the length of a vector, e.g. $\langle (3, 4), (3, 4) \rangle = 9 + 16 = 25 = |\alpha|^2$. The length of a vector α, also called the norm of α, will be denoted as $|\alpha|$. We get that:

$$|\alpha| = \sqrt{\langle \alpha, \alpha \rangle}.$$

5.2 Example

E.g.: $|(1, 2, 0, -2)| = \sqrt{1 + 4 + 0 + 4} = \sqrt{9} = 3$.

5.3 Exemplary problems

Find length (norm) of the following vectors: $(3, 4, 0)$, $(1, 1, 1, 0, -1)$, $(2, -2, 3, -2)$.

6 Calculating angle between vectors

6.1 Method

Assume now that we are given three vectors p, q and r forming a triangle. So $r = p - q$. Let θ be the angle between p i q. The law of cosines states that:

$$|r|^2 = |p|^2 + |q|^2 - 2|p||q|\cos\theta.$$

Therefore:

$$\langle p - q, p - q\rangle = \langle p, p\rangle + \langle q, q\rangle - 2|p||q|\cos\theta,$$

So:

$$\langle p, p\rangle + \langle q, q\rangle - 2\langle p, q\rangle = \langle p, p\rangle + \langle q, q\rangle - 2|p||q|\cos\theta,$$

So cosine of an angle between vectors is given by the following formula:

$$\cos\theta = \frac{\langle p, q\rangle}{|p||q|}.$$

6.2 Example

Let us calculate the angle between $(1, \sqrt{3}, 0)$ and $(3, 0, 0)$.

We get

$$\cos\theta = \frac{3}{2 \cdot 3} = 1/2,$$

so the angle is $60°$.

6.3 Exemplary problems

1. Calculate $(1, \sqrt{3}, 0)$ and $(0, 4, 0)$.
2. Let $v = (1, 1, 1, 1)$. Find an example of vector w such that the angle between v and w equals $45°$.

7 Finding a projection of a vector onto a linear subspace

7.1 Method

One more application of scalar product is calculating perpendicular projection of a vector onto a direction given by a second vector. Let r be the perpendicular projection of p onto direction given by q. It will have the same direction as q and length $|p|\cos\theta$, where θ is the angle between p and q. Therefore:

$$r = |p|\cos\theta \cdot \frac{q}{|q|} = q \cdot \frac{\langle p, q\rangle}{\langle q, q\rangle},$$

because: $\frac{q}{|q|}$ is the vector of length 1 in direction of q.

We already know how to calculate a projection of a vector onto a line defined by a given vector. To calculate a projection onto more-dimensional subspace V we have to calculate its orthogonal basis and calculate projections onto directions defined by each of the vectors from the basis and sum those projections.

Notice also that if r is a projection of a vector v onto V, and r' is its projection onto $V^{\perp}$, then $r = v - r'$. Which can be nicely used when calculating a projection of a vector onto a plane in the case of three-dimensional space. Indeed, if V is a plane in $\mathbb{R}^3$, then the perpendicular space is a line and is given by a vector. Denote this vector by n. Therefore the projection of v onto plane V is:

$$r = v - \frac{\langle v, n\rangle}{\langle n, n\rangle} n.$$

7.2 Example

The projection of $v = (1, 0, 1)$ onto line $\mathrm{lin}((1, 2, 3))$ is

$$\frac{\langle (1, 0, 1), (1, 2, 3)\rangle}{\langle (1, 2, 3), (1, 2, 3)\rangle}(1, 2, 3) = \frac{4}{14}(1, 2, 3) = \frac{2}{7}(1, 2, 3).$$

The projection of v onto plane $x + 2y + 3z = 0$ is

$$\frac{(1, 0, 1) - \langle (1, 0, 1), (1, 2, 3)\rangle}{\langle (1, 2, 3), (1, 2, 3)\rangle}(1, 2, 3) = (1, 0, 1) - \frac{4}{14}(1, 2, 3) = (1, 0, 1) - \frac{2}{7}(1, 2, 3). = \frac{1}{7}(5, -4, 1).$$

7.3 Exemplary problems

Calculate the projection of $(1, 0, 1, 0)$ onto

- the line $\mathrm{lin}((1, 1, 1, 1))$,
- the hyperplane described by $x + y + z + t = 0$,
- the plane $\mathrm{lin}((1, 1, 0, 0), (0, 1, 1, 0))$ (notice that you have to find an orthogonal basis here).

8 Finding a basis of the perpendicular space given system of vectors spanning a linear space

8.1 Method

We know that two vectors v, w are perpendicular if cosine of the angle between them equals zero. Therefore $v \perp w$ if and only if $\langle v, w \rangle = 0$.

Notice that if we would like to find all vectors w perpendicular to v, then the above is the equation we have to solve. Moreover this is a linear uniform equation. If we would like to find the set of vectors perpendicular to all vectors from a given list, then we will get a system of uniform linear equations. So given a linear subspace V, the set $V^\perp$ of all vectors perpendicular to all the vectors from V is also a linear subspace! It is the space of solutions of some system of linear equations.

Let $V = lin(v_1, v_2, \ldots)$. Then $(x, y, \ldots)$ is perpendicular to those vectors (thus to all vectors from V), if $\langle v_1, (x, y, \ldots) \rangle = 0$, $\langle v_2, (x, y, \ldots) \rangle = 0$, so if it satisfies a system on equations, the rows of matrix of which are $v_1, v_2, \ldots$. The system has to be solved to obtain a basis.

Notice that the coefficients in a system of equations describing given linear space are vectors which span the perpendicular space! Which is a new insight into our method of finding a system of equations for a space given by its spanning vectors.

8.2 Example

For example, let $V = lin((1, 1, 0, -1), (-1, 0, 2, 0))$. A vector (x, y, z, t) is perpendicular to those vectors (and so also to every vector of V), if $\langle (1, 1, 0, -1), (x, y, z, t) \rangle = 0$ and $\langle (-1, 0, 2, 0), (x, y, z, t) \rangle = 0$, in other words, if it satisfies the following system of equations:

$$\begin{cases} x + y - t = 0 \\ -x + 2z = 0 \end{cases},$$

so:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & 0 \end{array}\right] \xrightarrow{w_2 + w_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{w_1 - w_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

So the general solution has the following form: $(2z, -2z + t, z, t)$, and therefore we have the following basis of $V^\perp$: $(2, -2, 1, 0), (0, 1, 0, 1)$.

8.3 Exemplary problems

Find a basis of $V^\perp$, if

- $V = \text{lin}((1, 1, 1)) \subseteq \mathbb{R}^3$,
- $V = \text{lin}((1, 2, 0, 1), (0, 1, 1, 1), (3, 3, -1, 0)) \subseteq \mathbb{R}^4$.

9 Finding a basis of the perpendicular space given system of equations describing a linear space

9.1 Method

Therefore, it is even easier to get a system spanning the perpendicular space being given a system of equations spanning the space. It suffices to take the vectors with coefficients taken from the subsequent equations.

9.2 Example

Let $V \subseteq \mathbb{R}^4$ be described by

$$\begin{cases} x + 2y - t = 0 \\ 2y + z + 3t = 0 \end{cases},$$

to $V^\perp = \text{lin}((1,2,0,-1),(0,2,1,3))$.

9.3 Exemplary problems

1. Find a basis of $V^\perp$, if $V \subseteq \mathbb{R}^3$ is described by $x + 2y - z = 0$.
2. Find a basis of $V \subseteq \mathbb{R}^5$, if $V^\perp$ is described by

$$\begin{cases} x + y + z + t + w = 0 \\ x + y - w = 0 \end{cases}.$$

3. Find a system of equations for V, if $V^\perp = \text{lin}((1,1,0),(2,1,-1))$.

10 Finding coordinates of a vector with respect to a given orthogonal basis

10.1 Method

A basis of a space will be called orthogonal, if every pair of vectors in it are perpendicular.

We will learn how to find such basis later on. Now let us notice that calculating coordinates of a vector in such a basis is fairly simple. Actually, we are calculating its projections onto vectors from the basis. Given vector v, its n-th coordinate in an orthogonal basis, which has b_n as its n-th vector is simply $\frac{\langle v, b_n \rangle}{\langle b_n, b_n \rangle}$.

10.2 Example

E.g.

$$(1, 0, 1), (-1, 1, 1), (-1, -2, 1)$$

is an orthogonal basis of $\mathbb{R}^3$ – indeed it is easy to check, that all pairs are perpendicular, e.g.

$$\langle (-1, 1, 1), (-1, -2, 1) \rangle = 1 - 2 + 1 = 0.$$

Therefore, coordinates of $(1, 0, -2)$ in the above exemplary basis are: $\frac{\langle (1,0,-2),(1,0,1) \rangle}{\langle (1,0,1),(1,0,1) \rangle} = \frac{-1}{2}$, $\frac{-3}{3} = -1$ and $\frac{-3}{6} = \frac{-1}{2}$.

10.3 Exemplary problems

Find the coordinates of $(2017, 2, 1, 2)$ with respect to the orthogonal basis

$$(-1, 0, 0, 0), (0, 2, 2, 1), (0, -1, 1, 0), (0, 1, 1, -4).$$

11 Finding coordinates of a vector with respect to a given orthogonal basis

11.1 Method

Orthonormal basis is an orthogonal basis in which additionally all the vectors have length 1. Notice that calculating the coordinates of a vector in an orthonormal basis is even simpler. Since $\langle b_n, b_n\rangle = 1$, we get that the n-th coordinate of v is simply $\langle v, b_n\rangle$, where b_n is the n-th vector of the basis.

11.2 Example

So $\frac{(1,0,1)}{\sqrt{2}}, \frac{(-1,1,1)}{\sqrt{3}}, \frac{(-1,2,1)}{\sqrt{6}}$ is an orthonormal basis of $\mathbb{R}^3$.

Therefore the coordinates of $(1, 0, -2)$ in the basis in the example are: $\frac{-1}{\sqrt{2}}$, $\frac{-3}{\sqrt{3}}$ and $\frac{-3}{\sqrt{6}}$.

11.3 Exemplary problems

Find the coordinates of $(2017, 2, 1, 2)$ with respect to orthonormal basis

$$(-1, 0, 0, 0), (0, 2/3, 2/3, 1/3), (0, -1, 1, 0)/\sqrt{2}, (0, 1/3, 1/3, -4/3)/\sqrt{2}.$$

12 Finding an orthogonal basis

12.1 Method

Assume that we are given a space spanned by some vectors, e.g. $V = lin(v_1, v_2, v_3)$. We would like to find an orthogonal basis of this space. The method of finding such a basis is called Gram-Schmidt orthogonalization. The idea is to take as the first vector of the new basis the first vector from the original basis:

$$w_1 = v_1.$$

The second vector has to be be perpendicular to the first one, so we take the second vector and subtract its projection onto the first one. So:

$$w_2 = v_2 - \frac{\langle w_1, v_2\rangle}{\langle w_1, w_1\rangle} w_1$$

so only perpendicular „part" is left. In the case of the third vector we need to subtract projections onto both already constructed vectors:

$$w_3 = v_3 - \frac{\langle w_1, v_3\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle w_2, v_3\rangle}{\langle w_2, w_2\rangle} w_2.$$

In the case of more dimensions we will continue this procedure further.

12.2 Example

Let $V = lin((1, 0, 1, 0), (0, 1, -1, 1), (0, 0, 0, 1))$ and let us find an orthogonal basis.

$$w_1 = (1, 0, 1, 0).$$

$$w_2 = (0, 1, -1, 1) - \frac{\langle (1, 0, 1, 0), (0, 1, -1, 1)\rangle}{\langle (1, 0, 1, 0), (1, 0, 1, 0)\rangle}(1, 0, 1, 0) =$$
$$= (0, 1, -1, 1) - \frac{-1}{2}(1, 0, 1, 0) = \frac{1}{2}(1, 2, -1, 2).$$

$$w_3 = (0, 0, 0, 1) - \frac{\langle (1, 0, 1, 0), (0, 0, 0, 1)\rangle}{\langle (1, 0, 1, 0), (1, 0, 1, 0)\rangle}(1, 0, 1, 0)+$$
$$-\frac{\langle (1, 2, -1, 2), (0, 0, 0, 1)\rangle}{\langle (1, 2, -1, 2), (1, 2, -1, 2)\rangle}(1, 2, -1, 2) =$$
$$= (0, 0, 0, 1) - \frac{0}{2}(1, 0, 1, 0) - \frac{2}{10}(1, 2, -1, 2) = \frac{1}{5}(-1, -2, 1, 3).$$

So the basis we were looking for is $(1, 0, 1, 0), (1, 2, -1, 2), (-1, -2, 1, 3)$ (I can drop the fractions since multiplication by a number does not change angles).

12.3 Exemplary problem

Let $V = \mathrm{lin}((1, 1, 0, 0, 0), (0, 2, 1, 0, 0), (-1, 0, -1, 1, 1)$. Find an orthogonal basis of V and orthogonal basis of $V^\perp$.

13 Finding an orthonormal basis

13.1 Method

To find an orthonormal basis you have to find an orthogonal basis first. To transform an orthogonal basis into an orthonormal, you have to divide each vector by its length.

13.2 Example

We have previously found an orthogonal basis $(1, 0, 1, 0), (1, 2, -1, 2), (-1, -2, 1, 3)$. Thus,

$$(1, 0, 1, 0)/\sqrt{2}, (1, 2, -1, 2)/\sqrt{10}, (-1, -2, 1, 3)/\sqrt{15}$$

is an orthonormal basis of $\mathbb{R}^3$.

13.3 Exemplary problems

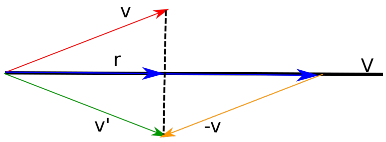
Let $V = \text{lin}((1, 1, 0, 0, 0), (0, 2, 1, 0, 0), (-1, 0, -1, 1, 1)$. Fin an orthonormal basis of V and an orthonormal basis of $V^{\perp}$.

14 Finding an image of a vector under reflection across linear subspace

14.1 Method

Now that we know how to calculate the projection r of v onto space V, it is easy (see the figure below) to calculate the image v' of v under reflection across V:

$$v' = 2r - v$$



14.2 Example

We have previously calculated that the projection of $v = (1, 0, 1)$ onto plane $x + 2y + 3z = 0$ equals $\frac{1}{7}(5, -4, 1)$.

So the image of v under the reflection across this plane is

$$\frac{2}{7}(5, -4, 1) - (1, 0, 1) = \frac{1}{7}(3, -8, -5).$$

14.3 Exemplary problems

Find the image of $(1, 0, 1, 0)$ under reflection across

- the line $lin((1, 1, 1, 1))$,
- the hyperplane described by $x + y + z + t = 0$,
- the plane $\text{lin}((1, 1, 0, 0), (0, 1, 1, 0))$.

15 Finding a formula for linear transformation of projection or reflection with respect to a linear subspace

15.1 Method

Notice that a projection onto a linear subspace V and symmetry with respect to V are linear mappings. Moreover, it is easy to see its eigenvectors:

- since projection does not change vectors in V, they are eigenvectors with eigenvalue 1. On the other hand, vectors from $V^{\perp}$ are multiplied by zero, so they are eigenvectors with eigenvalue zero.
- since symmetry does not change vectors in V, they are eigenvectors with eigenvalue 1. On the other hand, vectors from $V^{\perp}$ are multiplied by -1, so they are eigenvectors with eigenvalue -1.

So basis consisting of vectors from a basis of V and of vectors from a basis of $V^{\perp}$ is a basis of eigenvectors of both those maps. Which make it possible to calculate their formulas.

15.2 Example

E.g. let $V = \text{lin}((1,0,1),(0,1,-1))$. Therefore, basis $V^{\perp}$ is $\{(-1,1,1)\}$. So if ϕ is the projection onto V, and ψ is the symmetry with respect to V, then $(1,0,1),(0,1,-1)$ are eigenvectors with eigenvalue 1 of both maps. Also $(-1,1,1)$ is an eigenvector with eigenvalue zero for ϕ, and -1 for ψ. Therefore, basis

$$\mathcal{A} = \{(1,0,1),(0,1,-1),(-1,1,1)\}$$

is a basis of eigenvectors of both maps, and:

$$M(\phi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M(\psi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Let us calculate their formulas. We have:

$$M(id)_{st}^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

So:

$$M(id)_{\mathcal{A}}^{st} = \left(M(id)_{\mathcal{A}}^{st}\right)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ -1 & 1 & 1 \end{bmatrix},$$

Therefore:

$$M(\phi)_{st}^{st} = M(id)_{\mathcal{A}}^{st} M(\phi)_{\mathcal{A}}^{\mathcal{A}} M(id)_{st}^{\mathcal{A}} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and

$$M(\psi)_{st}^{st} = M(id)_{\mathcal{A}}^{st} M(\psi)_{\mathcal{A}}^{\mathcal{A}} M(id)_{st}^{\mathcal{A}} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

so:

$$\phi((x,y,z)) = \frac{1}{3}(2x+y+z, x+2y-z, x-y+2z)$$

and

$$\psi((x,y,z)) = \frac{1}{3}(x+2y+2z, 2x+y-2z, 2x-2y+z).$$

15.3 Exemplary problems

Find formulas for linear transformations φ and ψ of projection and reflection with respect to:

- the line $lin((1, 1, 1, 1))$,
- the hyperplane described by $x + y + z + t = 0$,
- the plane $\text{lin}((1, 1, 0, 0), (0, 1, 1, 0))$.

16 Finding a direction and translation vector given points which an affine subspace goes through

16.1 Method

Till now we have dealt with linear spaces, in particular in all our lines, planes and other subspaces the zero vector was always included. But obviously it makes sense to study also such spaces but translated from zero by a given vector. Such subspaces (are no longer linear) are called affine subspaces of a linear space. The term hyperplane is used to describe affine subspaces of dimension one less then the whole linear space we work in.

So if V is a linear subspace and v is a vector, then $M = v + V$ (meaning subspace V translated by v), is an affine subspace. The space V will be called the tangent space (or the direction) to M and denoted by $T(M)$ (or $\vec{M}$).

Given a translation vector and the tangent space, we can easily get the points, which are included in the considered affine subspace. E.g. if $H = p + lin(v, w)$, it can be also defined as the only affine subspace including points: $p, v + p, w + p$.

Reversely, if H goes through: p, q, r, choose one of those points as a translation vector, and we get that $H = p + lin(q - p, r - p)$.

16.2 Example

E.g. if $H = (1, 0, 0, -1) + lin((1, -1, 0, 1), (2, -1, 1, 0))$, it can be also defined as the only affine subspace including points: $(1, 0, 0, -1), (1, 0, 0, -1) + (1, -1, 0, 1) = (2, -1, 0, 0), (1, 0, 0, -1) + (2, -1, 1, 0) = (3, -1, 1, -1)$.

Reversely, if H goes through: $(1, 0, 0, -1), (2, -1, 0, 0), (3, -1, 1, -1)$, choose one of those points as a translation vector, and we get that $H = (1, 0, 0, -1) + lin((2, -1, 0, 0) - (1, 0, 0, -1), (3, -1, 1, -1) - (1, 0, 0, -1)) = (1, 0, 0, -1) + lin((1, -1, 0, 1), (2, -1, 1, 0))$.

16.3 Exemplary problems

1. Give an example of three points belonging to $(1, 0, 1) + \text{lin}((1, 2, 2), (-3, 2, 1))$.

2. Find a vector of translation and the direction of the hyperplane going through

$$(1, 2, 3, 0), (1, 0, 0, 1), (1, -2, 1, 0), (1, 1, 0, -1).$$

17 Finding a parametrization of an affine subspace given its direction and translation vector

17.1 Method

Given a translation vector and vectors spanning the tangent space we can easily get a parametrization. If $H = p + lin(v, w)$, then every vector of H is of form $p + tv + sw$, $t, s \in \mathbb{R}$ which gives us a parametrization.

17.2 Example

Given a translation vector and vectors spanning the tangent space we can easily get a parametrization. E.g. if $H = (1, 0, 0, -1) + lin((1, -1, 0, 1), (2, -1, 1, 0))$, then every vector of H is of form $(1, 0, 0, -1) + t(1, -1, 0, 1) + s(2, -1, 1, 0)$, so $H = \{(1 + t + 2s, -t - s, s, -1 + t) \colon s, t \in \mathbb{R}\}$.

Reversely, given $H = \{(1 + t + 2s, -t - s, s, -1 + t) \colon s, t \in \mathbb{R}\}$ we know that all the vectors of H are of form $(1, 0, 0, -1) + t(1, -1, 0, 1) + s(2, -1, 1, 0)$, so $H = (1, 0, 0, -1) + lin((1, -1, 0, 1), (2, -1, 1, 0))$.

17.3 Exemplary problems

1. Find a parametrization of $(1, 0, 1) + \text{lin}((1, 2, 2), (-3, 2, 1))$.
2. Find a parametrization of the hyperplane going through
$$(1, 2, 3, 0), (1, 0, 0, 1), (1, -2, 1, 0), (1, 1, 0, -1).$$
3. Fin a vector of translation and the direction of $\{(1 - x + 2y, x - y, 5 + 2x - 3y) \colon x, y \in \mathbb{R}\}$.

18 Finding a system of equations describing an affine subspace

18.1 Method

A system of linear equations describing a given affine subspace differ from the system of equation describing its tangent space only by the column of free coefficients (because the difference between two points from H always is in $T(H)$, so the difference of two solutions of the system of linear equations we are looking for is a solution of the system of linear equations describing the tangent space) – and in the second space we have an uniform system which we already know how to find. Therefore start with finding the system describing $T(H)$.

The given translation vector has to be a solution of the system we are looking for, so we have to choose the free coefficients in such a way that it will be true. So we can simply substitute the translation vector into the left sides of the equations to get the right hand side.

18.2 Example

For example, let $H = (1,0,0,-1) + lin((1,-1,0,1),(2,-1,1,0))$. First find the system describing $T(H) = lin((1,-1,0,1),(2,-1,1,0))$ – we already know how to find such a system and so $T(H)$ is given by:

$$\begin{cases} -x - y + z = 0 \\ x + 2y + w = 0 \end{cases}$$

Next, so we can simply substitute the translation vector into the left sides of the equations: $-1 - 0 + 0 = -1, 1 + 0 + -1 = 0$. Therefore, the system of equations we are looking for is the following:

$$\begin{cases} -x - y + z = -1 \\ x + 2y + w = 0 \end{cases}$$

18.3 Exemplary problems

1. Find a system of equations for $(1,0,1) + \mathrm{lin}((1,2,2),(-3,2,1))$.
2. Find a system of equations for the hyperplane going through

$$(1,2,3,0),(1,0,0,1),(1,-2,1,0),(1,1,0,-1).$$

3. Find a system of equations for $\{(1 - x + 2y, x - y, 5 + 2x - 3y) \colon x, y \in \mathbb{R}\}$.

19 Finding a parametrization of an affine subspace given a system of equations

19.1 Method

Notice that given a system for H, its general solution gives a parametrization for H!

19.2 Example

Let $V \subseteq \mathbb{R}^3$ be described by $x + y - 2z = 3$. The general solution takes form: $x = 3 - y + 2z$, thus we get parametrization $\{(3 - y + 2z, y, z) \colon y, z \in \mathbb{R}\}$.

19.3 Exemplary problems

Find parametrization of affine spaces $M \subseteq \mathbb{R}^4$:

- described by the following system of equations

$$x + y - z + 2t = 12x - 2y - z = -2.$$

- of solutions to $x + y + z + t = 1$.

20 Finding an affine subspace perpendicular to a given affine subspace and going through a given point

20.1 Method

The key observation in solving such problem is that if $M \perp H$, then $\vec{M} \perp \vec{H}$. Hence, (if the dimensions sum up to the dimension of the whole space) $\vec{M} = (\vec{H})^{\perp}$. In other words, the coefficients of a system of equations describing $\vec{H}$ give a system of vectors spanning $\vec{M}$. Or the other way around, a system of vectors spanning $\vec{H}$ gives coordinates of a system of equations on $\vec{M}$.

20.2 Example

Let us find a system of equations describing plane M going through $(1, 2, -1)$ and perpendicular to line $L = (2017, 0, 0) + \text{lin}((1, 1, 1))$.

Then $(\vec{(M)})^{\perp} = \vec{L} = \text{lin}((1, 1, 1))$, so $\vec{M}$ is given by equation $x + y + z = 0$. Thus, M is described by $x + y + z = 2$ (we substitute $(1, 2, -1)$ to get the free coefficient).

20.3 Exemplary problems

Find a system of equations an a parametrization of

- the line L going through $(1, 0, 0, 0)$ and perpendicular to the hyperplane

$$H = \{(1 + x - y + z, 2x + z, y - 3x, -1) \colon x, y, z \in \mathbb{R}\},$$

- the plane M going through $(1, -1, 1, -1)$ and perpendicular to the plane described by

$$\begin{cases} x + 3y + t = 0 \\ 2x + 7y - z - t = 0 \end{cases} .$$

21 Finding a projection of a point or its image under reflection with respect to an affine subspace

21.1 Method

How to calculate a projection of a vector onto an affine subspace and its image under reflection with respect to such a subspace? Simply bring it to the know case of linear subspaces, calculate, and go back to the initial setting. In other words translate everything to make the considered affine subspace go through zero, calculate the projection and translate everything backwards.

21.2 Example

Let us calculate the projection of $(2, 2, 1)$ onto $(2, 1, 0) + lin(-1, -1, 0)$. First we calculate the projection of $(2, 2, 1) - (2, 1, 0) = (0, 1, 1)$ onto $lin(-1, -1, 0)$:

$$\frac{\langle(0, 1, 1), (-1, -1, 0)\rangle}{\langle(-1, -1, 0), (-1, -1, 0)\rangle}(-1, -1, 0) = \frac{-1}{2}(-1, -1, 0) = \left(\frac{1}{2}, \frac{1}{2}, 0\right).$$

And translate this projection backwards, so the final result is: $\left(\frac{1}{2}, \frac{1}{2}, 0\right) + (2, 1, 0) = \left(\frac{5}{2}, \frac{3}{2}, 0\right)$.

21.3 Exemplary problems

1. Find the projection of $(1, 0, 1, 0)$ onto
   - the line $(1, 0, -1, 0) + \text{lin}((1, 1, 1, 1))$,
   - the hyperspace described by $x + y + z + t = -3$,
   - the plane $af((1, 1, 0, 0), (0, 1, 1, 0), (1, 1, 1, 1))$.
2. Find the image of $(1, 0, 1, 0)$ under reflection across
   - the line $(1, 0, -1, 0) + \text{lin}((1, 1, 1, 1))$,
   - the hyperspace described by $x + y + z + t = -3$,
   - the plane $af((1, 1, 0, 0), (0, 1, 1, 0), (1, 1, 1, 1))$.