Linear algebra , WNE, 2017/2018
exemplary problems 3. test

21 grudnia 2017

Problem 1.

Calculate by finding eigenvectors:

$$\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{16}{3} & \frac{2}{3} & \frac{16}{3} \\ -\frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} & \frac{4}{3} & \frac{2}{3} \\ 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & -6 & 1 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{2017}$$

Solution: We start by calculating the characteristic polynomial

$$w(\lambda) = ((\frac{1}{3}-\lambda)(-\frac{1}{3}-\lambda)-\frac{8}{9})(-1-\lambda)(1-\lambda)(1-\lambda) = (\lambda^2-1)(-1-\lambda)(1-\lambda)(1-\lambda) = -(\lambda-1)^3(\lambda+1)^2.$$

So the eigenvalues are ± 1.
Consider $\lambda = -1$:

$$V_{(-1)}\colon \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & -\frac{16}{3} & \frac{2}{3} & \frac{16}{3} \\ -\frac{4}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{4}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -6 & 2 & 6 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} -2 & 1 & 5 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we get a basis of $V_{(-1)}$: $\{(1,2,0,0,0),(0,-5,1,3,0)\}$.
Consider $\lambda = 1$:

$$V_{(1)}\colon \begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} & -\frac{16}{3} & \frac{2}{3} & \frac{16}{3} \\ -\frac{4}{3} & -\frac{4}{3} & -\frac{2}{3} & \frac{4}{3} & \frac{2}{3} \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & -6 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we get a basis of $V_{(1)}$: $\{(-1,1,0,0,0),(1,0,0,1,0),(0,0,1,0,1)\}$, and putting them together we get

$$\mathcal{A} = \{(1,2,0,0,0),(0,-5,1,3,0),(-1,1,0,0,0),(-1,0,0,1,0),(0,0,1,0,1)\}.$$

Let φ be such that $M(\varphi)_{st}^{st}$ is the matrix given in the problem. Then

$$M(\varphi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

So finally,

$$M(\phi)_{\mathrm{st}}^{\mathrm{st}})^{2017} = M(\mathrm{id})_{\mathcal{A}}^{\mathrm{st}} \cdot \begin{bmatrix} (-1)^{2017} & 0 & 0 & 0 & 0 \\ 0 & (-1)^{2017} & 0 & 0 & 0 \\ 0 & 0 & 1^{2017} & 0 & 0 \\ 0 & 0 & 0 & 1^{2017} & 0 \\ 0 & 0 & 0 & 0 & 1^{2017} \end{bmatrix} \cdot M(\mathrm{id})_{\mathrm{st}}^{\mathcal{A}} =$$

$$= M(\mathrm{id})^{\mathrm{st}}_{\mathcal{A}} \cdot \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot M(\mathrm{id})^{\mathcal{A}}_{\mathrm{st}} = M(\varphi)^{\mathrm{st}}_{\mathrm{st}} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{16}{3} & \frac{2}{3} & \frac{16}{3} \\ -\frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} & \frac{4}{3} & \frac{2}{3} \\ 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & -6 & 1 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Which means that the matrix does not change.

Problem 2.

In $\mathbb{R}^4$, let $V = \mathrm{lin}(1, 2, -1, -1)$. Find an orthonormal basis of $V^\perp$. Give an example of vectors $\alpha, \beta \in V^\perp$, such that the angle between them equals $60°$ and $||\alpha|| = ||\beta|| = 3$.

Solution:

Space $V^\perp$ is described by equation $x + 2y - z - t = 0$, so its basis is

$$\{(-2, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1)\}.$$

We apply Gramm-Schmidt procedure to this basis

$$v_1 = (-2, 1, 0, 0)$$

$$v_2 = (1, 0, 1, 0) - \frac{-2}{5}(-2, 1, 0, 0) = \frac{1}{5}(1, 2, 5, 0)$$

$$v_3 = (1, 0, 0, 1) + \frac{2}{5}(-2, 1, 0, 0) - \frac{1}{30}(1, 2, 5, 0) = \frac{1}{30}(5, 10, -5, 30) = \frac{1}{6}(1, 2, -1, 6).$$

After normalization, we get $\{\frac{1}{\sqrt{5}}(-2, 1, 0, 0), \frac{1}{\sqrt{30}}(1, 2, 5, 0), \frac{1}{\sqrt{42}}(1, 2, -1, 6)\}$.

It is easy to find angle $60°$ given an orthonormal basis. We shall simply take vectors v_1 and $v_1 + \sqrt{3}v_2$, but they have to be of length 3, so we take $\alpha = 3v_1 = \{\frac{3}{\sqrt{5}}(-2, 1, 0, 0)$ and $\beta = \frac{3}{2}(v_1 + \sqrt{3}v_2) = \frac{3}{2}(\frac{1}{\sqrt{5}}(-2, 1, 0, 0) + \frac{1}{\sqrt{10}}(1, 2, 5, 0) = \frac{3\sqrt{5}}{10}(-2 + \frac{\sqrt{2}}{2}, 1 + \sqrt{2}, \frac{5\sqrt{2}}{2}, 0)$.

Problem 3.

Let $V \subseteq \mathbb{R}^4$ be equal to $\mathrm{lin}((1, 2, 1, -1), (1, 1, 0, 1)$. Find

- a formula for linear transformation φ being the reflection across V,
- a formula for linear transformation ψ being the projection onto V,
- a basis consisting of eigenvectors of φ and matrix of this transformation in this basis.

Solution:

First we need a basis of $V^\perp$, so we solve the following system of equations

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

we get a basis of $V^\perp$: $\{(1, -1, 1, 0), (-3, 2, 0, 1)\}$. Notice, that

$$\varphi((1, 2, 1, -1)) = (1, 2, 1, -1), \varphi((1, 1, 0, 1)) = (1, 1, 0, 1),$$

$$\varphi((1, -1, 1, 0)) = -(1, -1, 1, 0), \varphi((-3, 2, 0, 1)) = -(-3, 2, 0, 1),$$

so basis $\mathcal{A} = \{(1, 2, 1, -1), (1, 1, 0, 1), (1, -1, 1, 0), (-3, 2, 0, 1)\}$ consists of eigenvectors of φ and

$$M(\varphi)^{\mathcal{A}}_{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

We also need $M(id)^{\mathcal{A}}_{\text{st}} = (M(\text{id})^{\text{st}}_{\mathcal{A}})^{-1}$:

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & -3 & 1 & 0 & 0 & 0 \\ 2 & 1 & -1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{17} & \frac{4}{17} & \frac{3}{17} & \frac{-5}{17} \\ 0 & 1 & 0 & 0 & \frac{5}{17} & \frac{3}{17} & \frac{-2}{17} & \frac{9}{17} \\ 0 & 0 & 1 & 0 & \frac{-1}{17} & \frac{-4}{17} & \frac{14}{17} & \frac{5}{17} \\ 0 & 0 & 0 & 1 & \frac{-4}{17} & \frac{1}{17} & \frac{5}{17} & \frac{3}{17} \end{array}\right].$$

Thus,

$$M(\varphi)^{\text{st}}_{\text{st}} = M(\text{id})^{\text{st}}_{\mathcal{A}} \cdot M(\varphi)^{\mathcal{A}}_{\mathcal{A}} \cdot M(\text{id})^{\mathcal{A}}_{\text{st}} =$$

$$= \begin{bmatrix} 1 & 1 & 1 & -3 \\ 2 & 1 & -1 & 2 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \cdot \frac{1}{17} \begin{bmatrix} 1 & 4 & 3 & -5 \\ 5 & 3 & -2 & 9 \\ -1 & -4 & 14 & 5 \\ -4 & 1 & 5 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & -1 & 3 \\ 2 & 1 & 1 & -2 \\ 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & -1 \end{bmatrix} \cdot \frac{1}{17} \begin{bmatrix} 1 & 4 & 3 & -5 \\ 5 & 3 & -2 & 9 \\ -1 & -4 & 14 & 5 \\ -4 & 1 & 5 & 3 \end{bmatrix} = \frac{1}{17} \begin{bmatrix} -5 & 14 & 2 & 8 \\ 14 & 5 & 8 & -2 \\ 2 & 8 & -11 & -10 \\ 8 & -2 & -10 & 11 \end{bmatrix}.$$

Hence, $\varphi((x, y, z, t)) = \frac{1}{17}(-5x + 14y + 2z + 8t, 14x + 5y + 8z - 2t, 2x + 8y - 11z - 10t, 8x - 2y - 10z + 11t)$.

Similarly we proceed with ψ but this time notice that

$$M(\varphi)^{\mathcal{A}}_{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence,

$$M(\psi)^{\text{st}}_{\text{st}} = M(\text{id})^{\text{st}}_{\mathcal{A}} \cdot M(\psi)^{\mathcal{A}}_{\mathcal{A}} \cdot M(\text{id})^{\mathcal{A}}_{\text{st}} = \begin{bmatrix} 1 & 1 & 1 & -3 \\ 2 & 1 & -1 & 2 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \frac{1}{17} \begin{bmatrix} 1 & 4 & 3 & -5 \\ 5 & 3 & -2 & 9 \\ -1 & -4 & 14 & 5 \\ -4 & 1 & 5 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \cdot \frac{1}{17} \begin{bmatrix} 1 & 4 & 3 & -5 \\ 5 & 3 & -2 & 9 \\ -1 & -4 & 14 & 5 \\ -4 & 1 & 5 & 3 \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 6 & 7 & 1 & 4 \\ 7 & 11 & 4 & -1 \\ 1 & 4 & 3 & -5 \\ 4 & -1 & -5 & 14 \end{bmatrix},$$

Thus, $\psi((x, y, z, t)) = \frac{1}{17}(6x + 7y + z + 4t, 7x + 11y + 4z - t, x + 4y + 3z - 5t, 4x - y - 5z + 14t)$.

Problem 4.

Find a parametrization and a system of equations describing plane $H \subseteq \mathbb{R}^4$ going through $(0, -1, 2, 0)$, and perpendicular to plane G going through $(1, 0, 1, 1), (1, 3, 1, 2), (1, 0, 0, 2)$.

Solution: $\vec{G} = \text{lin}((1, 3, 1, 2) - (1, 0, 1, 1), (1, 0, 0, 2) - (1, 0, 1, 1)) = \text{lin}((0, 3, 0, 1), (0, 0, -1, 1))$, so $\vec{G}^{\perp} = \vec{H}$ is described by

$$\begin{cases} 3b + d = 0 \\ -c + d = 0 \end{cases}$$

So all vectors of form $(a, \frac{-d}{3}, d, d)$ belong to $\vec{H}$, and therefore we get the following parametrization of H: $(a, \frac{-d}{3}, d, d) + (0, -1, 2, 0) = (a, -1 + \frac{-d}{3}, 2 + d, d)$. The free coefficients in the system of equations are $3(-1) + 0 = -3$ and $-2 + 0 = -2$, so we get the following system of equations:

$$\begin{cases} 3b + d = -3 \\ -c + d = -2 \end{cases}$$