Linear algebra, WNE, 2018/2019
meeting 22. – solutions

18 December 2018

1. Find the projection of $p = (1, 2, 1) \in \mathbb{R}^3$:
   - onto plane M described by equation $2x + 3y - z = 2$,
   - onto line $L = (3, 2, -1) + \mathrm{lin}((1, -1, 1))$

   - Plane M goes through $(0, 0, -2)$, so we translate the whole picture by $(0, 0, 2)$. Then M goes to $\vec{M}$, and p to $p' = (1, 2, 3)$ and we want to find the projection of p' onto $\vec{M}$. The perpendicular vector to $\vec{M}$ is $n = (2, 3, -1)$, so the projection of p' onto $\vec{M}$ is $r' = p' - \frac{\langle p', n\rangle}{\langle n, n\rangle} n = (1, 2, 3) - \frac{5}{14}(2, 3, -1) = \frac{1}{14}(4, 13, 47)$. To get r, we have to translate r' back, so $r = r' - (0, 0, 2) = \frac{1}{14}(4, 13, 19)$.
   - We translate the whole picture by $(-3, -2, 1)$, so we have to calculate the projection of $(1, 2, 1) + (-3, -2, 1) = (-2, 0, 2)$ onto $\mathrm{lin}((1, -1, 1))$, and we get $\frac{\langle(-2,0,2),(1,-1,1)\rangle}{\langle(1,-1,1),(1,-1,1)\rangle}(1, -1, 1) = (0, 0, 0)$. So the answer is $(0, 0, 0) - (-3, -2, 1) = (3, 2, -1)$.

2. Find a system of equation and a parametrization of
   - the line in $\mathbb{R}^3$ going through $(3, 3, 4), (1, 2, 3)$,
   - the plane going through $(3, 0, 1), (2, 1, 0), (1, 1, 1)$.
   - the hyperplane going through $(3, 2, 1, -1)$ and perpendicular to the hyperplane described by the following system of equations
   $$\begin{cases} a + b + c + d = -9 \\ a - b + c + 2d = -4 \end{cases}$$

   - the line is $(3, 3, 4) + \mathrm{lin}((-2, -1, -1))$, so we get a parametrization $(3 - 2t, 3 - t, 4 - t)$. To find a system of equations, first we have to solve $-2a - b - c = 0$. We get basis $\{(-2, 1, 0), (-2, 0, 1)\}$. The free coefficients are $-6 + 3 = -3$ and $-6 + 4 = -2$, so the system we are looking for is:
   $$\begin{cases} -2x + y = -3 \\ -2x + z = -2 \end{cases}$$
   - this plane is $(3, 0, 1) + \mathrm{lin}((-1, 1, -1), (-2, 1, 0))$, so we get a parametrization $(3 - t - 2s, t + s, 1 - t)$. To find a system of equations, first we have to solve the following system of equations
   $$\begin{bmatrix} -1 & 1 & -1 \\ -2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$
   and we get basis $\{(1, 2, 1)\}$. The free coefficient is $3 + 1 = 4$, so we get equation $x + 2y + z = 4$.
   - thus the hyperplane we are asked about is $(3, 2, 1, -1) + \mathrm{lin}((1, 1, 1, 1), (1, -1, 1, 2))$, and we get a parametrization $(3 + t + s, 2 + t - s, 1 + t + s, -1 + t + 2s)$. To find a system we have to solve the following system
   $$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$
   and we get basis $\{(-3, 1, 0, 2), (-1, 0, 1, 0)\}$. The free coefficients are $-9+2-2 = -9$ and $-3+2 = -2$, so we get a system of equations:
   $$\begin{cases} -3x + y + 2t = -9 \\ -x + z = -2 \end{cases}$$

3. Find a projection of $p = (2,3,1) \in \mathbb{R}^3$:

   - onto plane M described by equation $x - 2y + z = 2$,
   - onto line $L = (-1,1,0) + \text{lin}((1,0,1))$

   - The plane M goes through $(0,0,2)$, so we translate the whole picture by $(0,0,-2)$. We have to find a projection of $(2,3,-1)$ onto the plane described by $x - 2y + z = 0$. This projection equals

   $$(2,3,-1) - \frac{\langle (2,3,-1),(1,-2,1)\rangle}{\langle (1,-2,1),(1,-2,1)\rangle}(1,-2,1) = (2,3-1) + \frac{5}{6}(1,-2,1) = \frac{1}{6}(17,8,-1)$$

   We translate everything back by $(0,0,2)$, and so we get $\frac{1}{6}(17,8,11)$.
   - We translate the whole picture by $-(-1,1,0) = (1,-1,0)$. Hence, we have to calculate the projection of $(3,2,1)$ onto line $\text{lin}((1,0,1))$. We get

   $$\frac{\langle (3,2,1),(1,0,1)\rangle}{\langle (1,0,1),(1,0,1)\rangle}(1,0,1) = \frac{4}{2}(1,0,1) = (2,0,2).$$

   Translating everything back by $(-1,1,0)$ we get the point we are looking for, which is $(1,1,2)$.

4. Calculate $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix}^{2017}$.

$$w(\lambda) = (1-\lambda)(3-\lambda)(2-\lambda)$$

so the eigenvalues are $1,2,3$. We consider the eigenspaces

$$V_{(1)} : \begin{bmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 2 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we get a basis of $V_{(1)}$: $\{(1,0,-2)\}$.

$$V_{(2)} : \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we get a basis of $V_{(2)}$: $\{(0,0,1)\}$.

$$V_{(3)} : \begin{bmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & -4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

we get a basis of $V_{(3)}$ $\{(1,1,-2)\}$.
Therefore, basis $\mathcal{A} = \{(1,0,-2),(0,0,1),(1,1,2)\}$ consists of eigenvectors.

Notice, that $M(\text{id})_{\text{st}}^{\mathcal{A}} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. So

$$A^{2017} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ -2 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{2017} & 0 \\ 0 & 0 & 3^{2017} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 2+3^{2017} & 0 \\ 0 & 3^{2017} & 0 \\ -2-2^{2017} & -4-2\cdot 3^{2017} & 2^{2017} \end{bmatrix}$$

5. Let $W \subseteq \mathbb{R}^3$ be described by $x - y + z = 0$. Find a formula for linear transformation of reflection across $W^\perp$.

Basis of W is $\{(1,1,0),(-1,0,1)\}$. And basis of $W^\perp$ is $\{(1,-1,1)\}$.

Notice that if φ is the reflection, then $\varphi((1,1,0)) = (-1,-1,0)$, $\varphi((-1,0,1)) = (1,0,-1)$ and $\varphi((1,-1,1)) = (1,-1,1)$, so $\mathcal{A} = \{(1,1,0),(-1,0,1),(1,-1,1)\}$ is a basis consisting of eigenvectors of φ with eigenvalues $-1,-1$ and 1 respectively.

We also have $M(\mathrm{id})^{\mathcal{A}}_{\mathrm{st}} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$. So

$$M(\varphi)^{\mathrm{st}}_{\mathrm{st}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} =$$

$$= \frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ -2 & -1 & -2 \\ 2 & -2 & -1 \end{bmatrix}$$

and hence, $\varphi((x,y,z)) = \frac{1}{3}(-x - 2y + 2z, -2x - y - 2z, 2x - 2y - z)$.