# Linear algebra, WNE, 2018/2019
# meeting 20. – solutions

11 December 2018

1. Consider in $\mathbb{R}^4$ with the standard scalar product the subspaces $V = \{(x, y, z, t) : x - y + 4z + 5t = 0\}$ and $W = \text{lin}((1, 0, -1, 2), (1, 1, 1, 1))$. Find orthonormal bases of $V^\perp$ and $W^\perp$.

   Basis $V^\perp$ has only one vector and it is $\{(1, -1, 4, 5)\}$, so to get an orthonormal basis we have to divide this vector by its length $||(1, -1, 4, 5)|| = \sqrt{2 + 16 + 25} = \sqrt{43}$, so the orthonormal basis is $\{\frac{1}{\sqrt{43}}(1, -1, 4, 5)\}$. $W^\perp$ is the space of solutions to the following system of equations.

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix},$$

   so we get basis $\{(1, -2, 1, 0), (-2, 1, 0, 1)\}$. Using Gram-Schmidt procedure

$$v_1 = (1, -2, 1, 0)$$

$$v_2 = (-2, 1, 0, 1) - \frac{-4}{6}(1, -2, 1, 0) = \frac{1}{3}(-4, -1, 2, 3)$$

   After normalization we get: $\{\frac{1}{\sqrt{6}}(1, -2, 1, 0), \frac{1}{\sqrt{30}}(-4, -1, 2, 3)\}$.

2. Check whether in $\mathbb{R}^4$ there exists a vector $\alpha$, such that $\frac{1}{2}(1, 1, 1, 1), \frac{1}{2}(-1, -1, 1, 1), \frac{1}{2}(-1, 1, -1, 1), \alpha$ is an orthonormal basis and the forth coordinate of $(2, 4, 6, 2)$ in this basis equals 3.

   Let $\alpha = (a, b, c, d)$. Those conditions give the following equations:

$$\begin{cases} a + b + c + d = 0 \\ -a - b + c + d = 0 \\ -a + b - c + d = 0 \\ a^2 + b^2 + c^2 + d^2 = 1 \\ 2a + 4b + 6c + 2d = 3 \end{cases}.$$

   We can solve the system consisting of the first, second, third and fifth equation in the usual way, and next check whether the solutions satisfies the forth equation as well. We get the solution: $\frac{1}{2}(-1, 1, 1, -1)$ and it actually satisfies the forth equation.

3. In $\mathbb{R}^3$, find the orthogonal projection of $\alpha = (1, 1, 1)$ onto $V = \{(x, y, z) : x + 2y - z = 0\}$ an the orthogonal projection of this vector onto line $L = \text{lin}((1, 2, 3))$.

   The space perpendicular to $V$ is one-dimensional and its basis is $(1, 2, -1)$. Therefore the orthogonal projection of $V$ onto $\alpha - \frac{\langle \alpha, (1,2,-1) \rangle}{||(1,2,-1)||^2}(1, 2, -1) = (1, 1, 1) - \frac{2}{6}(1, 2, -1) = \frac{1}{3}(2, 1, 4)$. Meanwhile the projection onto $L$ is simply $\frac{\langle (1,1,1),(1,2,3) \rangle}{||1,2,3||^2}(1, 2, 3) = \frac{3}{7}(1, 2, 3)$

4. Find the image of $\alpha$ from the previous problem under orthogonal reflection across $V$ and under orthogonal reflection across $L$.

   Analogously, reflection across $V$ gives: $(1, 1, 1) - \frac{2}{3}(1, 2, 1) = \frac{1}{3}(1, -1, 1)$.

   Reflection across $L$ is $\alpha + 2(\frac{3}{7}(1, 2, 3) - \alpha) = \frac{6}{7}(1, 2, 3) - (1, 1, 1) = \frac{1}{7}(-1, 5, 11)$.

5. In $\mathbb{R}^4$ find the formula for the linear transformation of orthogonal projection onto

$$W = \text{lin}((2,1,0,1),(1,0,0,1))$$

for the linear transformation of orthogonal reflection across $W$.

Let $\varphi$ be the projection, and $\psi$ be the reflection. We start by finding a basis of $W^{\perp}$:

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

so we get basis $\{(0,0,1,0),(-1,1,0,1)\}$. Notice that vectors

$$(2,1,0,1),(1,0,0,1),(0,0,1,0),(-1,1,0,1)$$

eigenvectors of $\varphi$ and $\psi$, because

- $\varphi((2,1,0,1)) = (2,1,0,1), \varphi((1,0,0,1)) = (1,0,0,1), \varphi((0,0,1,0)) = (0,0,0,0), \varphi((-1,1,0,1)) = (0,0,0,0)$,
- $\psi((2,1,0,1)) = (2,1,0,1), \psi((1,0,0,1)) = (1,0,0,1), \psi((0,0,1,0)) = -(0,0,1,0), \psi((-1,1,0,1)) = -(-1,1,0,1)$.

So considering basis $\mathcal{A} = \{(2,1,0,1),(1,0,0,1),(0,0,1,0),(-1,1,0,1)\}$, we get

$$M(\varphi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, M(\psi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The change-of-coordinates matrices are:

$$M(\text{id})_{\mathcal{A}}^{\text{st}} = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

and

$$M(\text{id})_{\text{st}}^{\mathcal{A}} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & \frac{-1}{3} \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{-1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

so

$$M(\varphi)_{\text{st}}^{\text{st}} = M(\text{id})_{\mathcal{A}}^{\text{st}} M(\varphi)_{\mathcal{A}}^{\mathcal{A}} M(\text{id})_{\text{st}}^{\mathcal{A}} = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & \frac{-1}{3} \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{-1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & \frac{-1}{3} \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{-1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 & \frac{-1}{3} \\ 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{-1}{3} & 0 & \frac{2}{3} \end{bmatrix}.$$

$$M(\psi)_{\text{st}}^{\text{st}} = M(\text{id})_{\mathcal{A}}^{\text{st}} M(\psi)_{\mathcal{A}}^{\mathcal{A}} M(\text{id})_{\text{st}}^{\mathcal{A}} = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & \frac{-1}{3} \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{-1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & \frac{-1}{3} \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{-1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 0 \\ 2 & -2 & 0 & 1 \end{bmatrix}.$$

Hence,

$$\varphi((a,b,c,d)) = \frac{1}{3}(2a+b+d, a+2b-d, 0, a-b+2d)$$

$$\psi((a,b,c,d)) = \frac{1}{3}(a+2b+2d, 2a+b-2d, -3c, 2a-2b+d).$$