Linear algebra, WNE, 2018/2019
meeting 20. – homework solutions

11 December 2018

Group 8:00

1. In $\mathbb{R}^4$, find the orthogonal projection of $(4,2,3,1)$ onto subspace $W=\{(x,y,z,t)\colon x+y-z+2t=0\}$ and the image of $(0,0,1,2)$ under orthogonal reflection across W.

   The projection onto W is the vector minus its projection onto $W^\perp$, i.e. $(1,1,-1,2)$, so

   $$(4,2,3,1)-\frac{\langle(4,2,3,1),(1,1,-1,2)\rangle}{\langle(1,1,-1,2),(1,1,-1,2)\rangle}(1,1,-1,2)=(4,2,3,1)-\frac{5}{7}(1,1,-1,2)=\frac{1}{7}(23,9,26,-3).$$

   Similarly the projection of $(0,0,1,2)$ equals to

   $$(0,0,1,2)-\frac{\langle(0,0,1,2),(1,1,-1,2)\rangle}{\langle(1,1,-1,2),(1,1,-1,2)\rangle}(1,1,-1,2)=(0,0,1,2)-\frac{3}{7}(1,1,-1,2)=\frac{1}{7}(-3,-3,10,8).$$

   So the image under reflection is

   $$\frac{1}{7}(-6,-6,20,16)-(0,0,1,2)=\frac{1}{7}(-6,-6,13,2).$$

2. Find the formula for the linear transformation $\varphi\colon\mathbb{R}^3\to\mathbb{R}^3$ of orthogonal projection onto

   $$W=\{(x,y,z)\colon x-y+2z=0\}$$

   for the linear transformation $\psi\colon\mathbb{R}^3\to\mathbb{R}^3$of orthogonal reflection across W.

   First notice that $\{(1,1,0),(-2,0,1)\}$ is a basis of W, and $\{(1,-1,2)\}$ is a basis of $W^\perp$. Let φ be the projection, and ψ be the reflection. $\mathcal{A}=\{(1,1,0),(-2,0,1),(1,-1,2)\}$ is a eigenvectors basis of φ and of ψ, with eigenvalues $1,1,0$ and $1,1,-1$ respectively. Thus,

   $$M(\varphi)^{\mathcal{A}}_{\mathcal{A}}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},M(\psi)^{\mathcal{A}}_{\mathcal{A}}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}.$$

   Meanwhile, $M(\mathrm{id})^{st}_{\mathcal{A}}=\begin{bmatrix}1&-2&1\\1&0&-1\\0&1&2\end{bmatrix}$, so it is easy to calculate

   $$M(\mathrm{id})^{\mathcal{A}}_{st}=(M(\mathrm{id})^{st}_{\mathcal{A}})^{-1}=\frac{1}{6}\begin{bmatrix}1&5&1\\-2&2&2\\1&-1&2\end{bmatrix}.$$

   Hence,

   $$M(\varphi)^{st}_{st}=M(\mathrm{id})^{st}_{\mathcal{A}}\cdot M(\varphi)^{\mathcal{A}}_{\mathcal{A}}\cdot M(\mathrm{id})^{\mathcal{A}}_{st}=\frac{1}{6}\begin{bmatrix}5&1&-2\\1&5&2\\-2&2&2\end{bmatrix}$$

   $$M(\psi)^{st}_{st}=M(\mathrm{id})^{st}_{\mathcal{A}}\cdot M(\psi)^{\mathcal{A}}_{\mathcal{A}}\cdot M(\mathrm{id})^{\mathcal{A}}_{st}=\frac{1}{3}\begin{bmatrix}2&1&-2\\1&2&2\\-2&2&1\end{bmatrix}$$

   and thus, $\varphi((x,y,z))=\frac{1}{6}(5x+y-2z,x+5y+2z,-2x+2y+2z),\psi((x,y,z))=\frac{1}{3}(2x+y-2z,x+2y+2z,-2x+2y+z)$.

Group 9:45

1. In $\mathbb{R}^4$, find the orthogonal projection of $(4,2,1,3)$ onto $W = \{(x,y,z,t)\colon x+y+2z-t=0\}$ and the image of $(0,0,2,1)$ under orthogonal reflection across W.

   The projection onto W is the vector minus its projection onto $W^\perp$, i.e. $(1,1,2,-1)$, so

   $$(4,2,1,3) - \frac{\langle (4,2,1,3),(1,1,2,-1)\rangle}{\langle (1,1,2,-1),(1,1,2,-1)\rangle}(1,1,2,-1) = (4,2,1,3) - \frac{5}{7}(1,1,2,-1) = \frac{1}{7}(23,9,-3,26).$$

   Similarly, the projection of $(0,0,2,1)$ is

   $$(0,0,2,1) - \frac{\langle (0,0,2,1),(1,1,2,-1)\rangle}{\langle (1,1,2,-1),(1,1,2,-1)\rangle}(1,1,2,-1) = (0,0,2,1) - \frac{3}{7}(1,1,2,-1) = \frac{1}{7}(-3,-3,8,10).$$

   So the reflection:

   $$\frac{1}{7}(-6,-6,16,20) - (0,0,2,1) = \frac{1}{7}(-6,-6,2,13).$$

2. Find a formula for the linear transformation $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ of the orthogonal projection onto $W = \{(x,y,z)\colon x+2y-z=0\}$, and of the linear transformation $\psi\colon \mathbb{R}^3 \to \mathbb{R}^3$ of orthogonal reflection across W.

   Notice, that $\{(-2,1,0),(1,0,1)\}$ is a basis of W, on the other hand $\{(1,2,-1)\}$ is a basis of $W^\perp$. Let φ be the projection, and ψ be the reflection. $\mathcal{A} = \{(-2,1,0),(1,0,1),(1,2,-1)\}$ is an eigenvectors basis of φ, and also of ψ with eigenvalues $1,1,0$ and $1,1,-1$, respectively so

   $$M(\varphi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, M(\psi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

   Meanwhile, $M(\mathrm{id})_{\mathcal{A}}^{st} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$, so it is easy to calculate that

   $$M(\mathrm{id})_{st}^{\mathcal{A}} = (M(\mathrm{id})_{\mathcal{A}}^{st})^{-1} = \frac{1}{6}\begin{bmatrix} -2 & 2 & 2 \\ 1 & 2 & 5 \\ 1 & 2 & -1 \end{bmatrix}.$$

   Therefore,

   $$M(\varphi)_{st}^{st} = M(\mathrm{id})_{\mathcal{A}}^{st} \cdot M(\varphi)_{\mathcal{A}}^{\mathcal{A}} \cdot M(\mathrm{id})_{st}^{\mathcal{A}} = \frac{1}{6}\begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

   $$M(\psi)_{st}^{st} = M(\mathrm{id})_{\mathcal{A}}^{st} \cdot M(\psi)_{\mathcal{A}}^{\mathcal{A}} \cdot M(\mathrm{id})_{st}^{\mathcal{A}} = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

   Hence, $\varphi((x,y,z)) = \frac{1}{6}(5x-2y+z, -2x+2y+2z, x+2y+5z), \psi((x,y,z)) = \frac{1}{3}(2x-2y+z, -2x-y+2z, x+2y+2z)$.