Linear algebra, WNE, 2017/2018
before the second test

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Contents

1 Checking whether a transformation is linear

1.1 Method

A linear map ϕ is a map which maps vectors from a given space V, to vectors from another linear space W ($\phi\colon V \to W$), and satisfying the linear condition, which says that for every vectors $v, v' \in V$ and numbers a, b we have $\phi(av + bv') = a\phi(v) + b\phi(v')$. E.g. a rotation around $(0, 0)$ is a linear map $\mathbb{R}^2 \to \mathbb{R}^2$. Given two vectors and digits we will get the same vector regardless of whether we rotate the vector first and then multiply by numbers and add them, or multiply by numbers, add and then rotate.

Therefore, to prove that a given map is a linear map we need to prove that for any two vectors and any two numbers it satisfies the linear condition.

Meanwhile, to disprove that a map is linear we need to find an example of two vectors along with two numbers such that the linear condition fail for them.

1.2 Example

Mapping $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ given as $\varphi(x, y) = -x + 2y$ is a linear map because if $(x, y), (x', y') \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$, then $\varphi(a(x, y) + b(x', y')) = \varphi((ax + bx', ay + by')) = -ax - bx' + 2ay + 2by' = a(-x + 2y) + b(-x' + 2y') = a\varphi((x, y)) + b\varphi((x', y'))$

On the other hand, $\psi\colon \mathbb{R} \to \mathbb{R}^2$ given as $\psi(x) = (2x + 1, 0)$ is not a linear map because $\psi(1 + 1) = \psi(2) = (5, 0) \neq (6, 0) = (3, 0) + (3, 0) = \psi(1) + \psi(1)$.

1.3 Exemplary problems

Check whether the following mappings are linear:

- $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi(x, y, z) = (z, y, x)$,
- $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^5$, $\varphi(x, y, z) = (z, 2z, 3z, 4z, 5z)$,
- $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^5$, $\varphi(x, y) = (z^2, x, y, x, y)$,
- $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$, $\varphi(x, y) = x + y$,
- $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^4$, $\varphi(x, y) = (0, x, y, 2x - y)$.

2 Finding the value of a given linear transformation on a vector given its formula

2.1 Method

Usually, we get a formula of a linear transformation. Then we simply substitute the given vector v into this formula to get the value $\varphi(v)$.

2.2 Example

For example, $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\psi((x,y)) = (y, -2x+y)$. Then to calculate the value for $(1,2)$, and we get $\psi((1,2)) = (2, -2+2) = (2,0)$.

2.3 Exemplary problem

Calculate $\varphi(v)$ for:

- $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi((x,y,z)) = (x+y, -x, 3y+z)$, $v = (1,1,2)$,
- $\varphi\colon \mathbb{R}^4 \to \mathbb{R}$, $\varphi((x,y,z,t)) = x - y + z - 2t$, $v = (1,-1,2,-2)$,
- $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^3$, $\varphi((x_1,x_2,x_3,x_4)) = (2017x_4, -x_3+x_2, 4x_2+x_3-2017x_4)$, $v = (2017,0,0,0)$.

3 Calculating a formula of a linear transformation given values of in on vectors from a basis

3.1 Method

Usually, linear maps will be given by their formulas. But sometimes we can define a linear map by giving its values on the vectors from a given basis only. This suffices to determine this map. First we have to calculate the coefficients of the standard basis in the given one (by solving a system of equations or by guessing).

It is so because

$$\varphi((x_1,\ldots,x_n)) = \varphi(x_1\varepsilon_1+\ldots+x_2\varepsilon_n) = x_1\varphi(\varepsilon_1)+\ldots+\varphi(\varepsilon_n),$$

where $\varepsilon_1,\ldots,\varepsilon_n$ are the vectors from the standard basis. If we get $\varphi(\varepsilon_1),\ldots\varphi(\varepsilon_n)$, we will have the formula.

How to calculate $\varphi(\varepsilon_1),\ldots\varphi(\varepsilon_n)$? We are given values of φ on some vectors, let that be $\varepsilon(\alpha_1),\ldots,\varepsilon(\alpha_n)$. Those vectors are always chosen in such a way that they constitute a basis. For all i, we can calculate the coordinates of ε_i, with respect to this basis:

$$\varepsilon_i = a_1\alpha_1+\ldots+a_n\alpha_n.$$

This can be done by guessing or by solving a system of linear equations (see the materials before the first test). Then,

$$\varphi(\varepsilon_i) = a_1\varphi(alpha_1)+\ldots+a_n\varphi(\alpha_n).$$

3.2 Example

E.g. let $\varphi\colon\mathbb{R}^2\to\mathbb{R}^3$ be such that $\varphi((1,-1)) = (1,2,1), \varphi((-1,0)) = (-1,0,1)$ (vectors $(1,-1)$, $(-1,0)$ constitute a basis of a plane). We see that $(1,0) = 0\cdot(1,-1)-1(-1,0)$ (coordinates: 0,-1) and $(0,1) = -(1,-1)-(-1,0)$ (coordinates: -1,-1). Thus, $\varphi((x,y)) = x\varphi((1,0))+y\varphi((0,1)) = x(-\varphi((-1,0)))+y(-\varphi((1,-1))-\varphi((-1,0))) = x(-(-1,0,1))+y(-(1,2,1)-(-1,0,1)) = x(1,0,-1)+y(0,-2,-2) = (x,-2y,-x-2y)$.

3.3 Exemplary problems

Calculate a formula for φ, if

- $\varphi\colon\mathbb{R}^3\to\mathbb{R}^2$, and $\varphi((2,1,0)) = (1,1)$, $\varphi((0,0,1)) = (5,-2)$ i $\varphi((2,0,1)) = (-10,0)$.
- $\varphi\colon\mathbb{R}^4\to\mathbb{R}$, and $\varphi((1,1,0,0)) = 1$, $\varphi((0,1,1,0)) = 2$, $\varphi((0,0,1,1)) = 3$ i $\varphi((1,0,0,1)) = 4$.
- $\varphi\colon\mathbb{R}^4\to\mathbb{R}^3$, $\varphi((2,1,1,1)) = (1,2,3)$, $\varphi((0,1,0,0)) = (3,2,1)$, $\varphi((0,0,1,0)) = (-1,-1,-1)$ and $\varphi((1,0,0,1)) = (0,0,0)$.

4 Addition of matrices and multiplying a matrix by a number. Multiplication of matrices

4.1 Method

To add two matrices they need to have the same size. We add matrices by their coefficients. Addition of matrices is associative and commutative, so for any matrices A, B, C, $A+B = B+A$ and $(A+B)+C = A+(B+C)$.¡/p¿

Multiplying a matrix by a scalar works simply by coefficients. Because of those operations it is easy to see that the set of matrices of given size is a linear space.

To multiply matrices, the first has to have the same number of columns as the second one of rows. The resulting matrix will have as many rows as the first one, and as many columns as the second one. We multiply the rows of the first matrix by the columns of the second in the sense that in the resulting matrix in a place in i-th row and j-th column we write the result of multiplication of i-th row of the first matrix with the j-th column of the second one, where by multiplication of row and column we mean multiplication of pairs of subsequent numbers summed up.

Matrix multiplication is associative, which means that for any A, B, C, $A \cdot (B \cdot C) = (A \cdot B) \cdot C$. But is not commutative! Notice that if the matrices are not square it is impossible to multiply them conversely. If they are square the result may be different.

4.2 Examples

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ -5 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-1 & 2+4 \\ 3-5 & -1+2 \\ 2+0 & 4+1 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -2 & 1 \\ 2 & 5 \end{bmatrix}.$$

$$2 \cdot \begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 2 \\ -2 & 4 & 6 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 \\ -5 & 2 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 \cdot (-1) + 3 \cdot (-5) + 1 \cdot 0 & 2 \cdot 4 + 3 \cdot 2 + 1 \cdot 1 \\ (-1) \cdot (-1) + 2 \cdot (-5) + 3 \cdot 0 & (-1) \cdot 4 + 2 \cdot 2 + 3 \cdot 1 \\ 0 \cdot (-1) + 1 \cdot (-5) + 0 \cdot 0 & 0 \cdot 4 + 1 \cdot 2 + 0 \cdot 1 \\ 1 \cdot (-1) + 0 \cdot (-5) + (-2) \cdot 0 & 1 \cdot 4 + 0 \cdot 2 + (-2) \cdot 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -17 & 13 \\ -9 & 3 \\ -5 & 2 \\ -1 & 2 \end{bmatrix}.$$

4.3 Exemplary problems

Calculate:

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix} \cdot \left((-2) \cdot \begin{bmatrix} -1 & 4 \\ -5 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -5 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

5 Writing our a matrix of a linear transformation in the standard bases from a formula, and a formula from a matrix in the standard bases

5.1 Method

Matrix of a linear transformation in the standard bases has the following property. When multiplied by a vector from the right we get the value on this vector. Therefore a matrix of a linear transformation has always the same number of columns as the dimension of the space of arguments, and the same number of rows as the dimension of the space of values.

Moreover, one can observe

$$M(\varphi)_s t^s t \cdot \varepsilon_i,$$

where ε_i is the i-th vector from the standard basis, equals to the i-th column of the matrix. Hence, this i-th column is $\varphi(\varepsilon_i)$. In other words, the coefficients in this column are exactly the coefficients by x_i in the formula. So, to write out the matrix in the standard bases one has to write out the coefficients related to subsequent variables in subsequent columns. Or we can write in rows the coefficients of subsequent coordinates of the value from the formula. And analogously, when we want to get a formula given a matrix in the standard bases.

So, if $\varphi((x_1, \ldots, x_n)) = (a_{1,1}x_1 + \ldots + a_{1,n}x_n, \ldots, a_{m,1}x_1 + \ldots + a_{m,n}x_n)$, then

$$M(\varphi)_{st}^{st} = \begin{bmatrix} a_{1,1} & \ldots & a_{1,n} \\ & \ldots & \\ a_{m,1} & \ldots & a_{m,n} \end{bmatrix}$$

5.2 Example

Let $\phi\colon \mathbb{R}^3 \to \mathbb{R}^2$, be such that $\phi(x, y, z) = (x - z, 2x + 3y - z)$. Then,

$$M(\phi)_{st}^{st} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \end{bmatrix}.$$

Let

$$M(\psi)_{st}^{st} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then $\psi\colon \mathbb{R}^2 \to \mathbb{R}^4$ and $\psi(x, y) = (x, 2x + 3y, -y, x + y)$.

5.3 Exemplary problems

Find matrix $M(\phi)_{st}^{st}$ for:

- $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$, $\varphi((x, y, z)) = (x + y, -x, 3y + z)$,
- $\varphi\colon \mathbb{R}^4 \to \mathbb{R}$, $\varphi((x, y, z, t)) = x - y + z - 2t$,
- $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^3$, $\varphi((x_1, x_2, x_3, x_4)) = (2017x_4, -x_3 + x_2, 4x_2 + x_3 - 2017x_4)$.

Find a formula for ϕ, if:

- $M(\phi)_{st}^{st} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$,
- $M(\phi)_{st}^{st} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$.

6 Operations on linear transformations

6.1 Method

Two linear transformation can be added (if they have the same spaces of arguments and spaces of variables), we can multiply a linear transformation by a number (if the space of values of one of them is the space of arguments of another).

Composition of linear variables means that we apply one of them and then we apply the second to the value. So $(\psi \circ \phi)(v) = (\psi(\phi(v))$.

6.2 First method: using formulas

In the case of adding an multiplying by a number, we simply add the formulas or multiply them by a number.

In the case of composition, we substitute one of the formulas into the second.

6.3 Second method: using matrices

This is a more general method, since we can manage not only the formulas (matrices in the standard bases), but also matrices in any bases. Adding the transformations translates into adding their matrices (the bases has to be the same), and multiplying a transformation by a number translates into multiplying the matrix by a number. Composing transformations means multiplying their matrices, but the bases have to agree in the following way.

$$M(\psi \circ \phi)_{\mathcal{A}}^{\mathcal{C}} = M(\psi)_{\mathcal{B}}^{\mathcal{C}} \cdot M(\phi)_{\mathcal{A}}^{\mathcal{B}}.$$

6.4 Example

If $\varphi((x, y)) = (-x, -2y), \psi((x, y)) = (0, 2x)$, to $(\varphi + 2\psi)((x, y)) = (-x, 4x - 2y)$. Calculating the same using matrices, we get:

$$M(\varphi + 2\psi)_{st}^{st} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix}.$$

If $\phi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is such that $\phi((x, y, z)) = (x - z, 2x + 3y - z)$ and $\psi\colon \mathbb{R}^2 \to \mathbb{R}^4$ is such that $\psi((a, b)) = (-b, a + b, -2a - b, 3a)$. Thus, $(\psi \circ \phi)((x, y, z)) = \psi(\phi((x, y, z)) = \psi(((x - z, 2x + 3y - z)) = (-2x - 3y + z, 3x + 3y - 2z, -4x - 3y + 3z, 3x - 3z)$. Calculating the same using matrices, we get

$$M(\psi \circ \phi)_{st}^{st} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \\ -2 & -1 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ 3 & 3 & -2 \\ -4 & -3 & 3 \\ 3 & 0 & -3 \end{bmatrix}.$$

6.5 Exemplary problems

1. Find a formula for $\psi \circ (\phi - 3\varphi)$, where $\psi\colon \mathbb{R}^4 \to \mathbb{R}^2$, $\psi((a, b, c, d)) = (2a + b, -a + c + 3d)$, $\phi, \varphi\colon \mathbb{R}^2 \to \mathbb{R}4$, $\phi((x, y)) = (x - y, 2y, -x, 2x)$, $\varphi((x, y)) = (-3x - y, x + 2y, 0, -2y)$.

2. Find $M(\psi \circ (\phi - 3\varphi))$, if $\psi\colon V \to W$, $\phi, \varphi\colon U \to V$ and $\mathcal{A}$ is a basis of U, $\mathcal{B}$ is a basis of V and $\mathcal{C}$ is a basis of W, and $M(\psi)_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $M(\phi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, $M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$.

7 Finding change-of-coordinates matrix (matrix of identity)

7.1 Method

There is a special linear map, which we call the identity, which does nothing. For example, in $\mathbb{R}^3$ it is $id((x,y,z)) = (x,y,z)$. Therefore given two basis $\mathcal{A}$ and $\mathcal{B}$, along with matrix $M(id)_{\mathcal{A}}^{\mathcal{B}}$ if we multiply this matrix from the right by the coordinates of a vector v in basis $\mathcal{A}$ we will get the coordinates also of v (as $id(v) = v$), but in basis $\mathcal{B}$. So matrix $M(id)_{\mathcal{A}}^{\mathcal{B}}$ changes the basis from $\mathcal{A}$ to $\mathcal{B}$.

Thus, in matrix $M(id)_{\mathcal{A}}^{\mathcal{B}}$ in the columns we put the coordinates of vectors from $\mathcal{A}$ with respect to $\mathcal{B}$. We can find those coordinates by guessing or solving systems of equations (such systems have the same left-hand side, so we can put them in one matrix).

Especially we will need matrices changing basis from the standard basis to the given one and from the given basis to the standard one. Let's check how to calculate them.

It is easy to calculate the matrix of basis change from a given basis to the standard basis. We will find $M(id)_{\mathcal{A}}^{st}$ (basis is defined in the example above). After multiplying it from the right by $1,0,0$ we will get its first column. On the other hand the first vector in $\mathcal{A}$ has in it coordinates 1,0,0, so the result of multiplication are the coordinates of this vector in the standard basis, so simply it is this vector. So simply the i-th column of this matrix is the i-th vector from the basis.

Now, the other case: from the standard basis to a given basis. We will calculate $M(id)_{st}^{\mathcal{B}}$. It can be easily seen that in columns we should put coordinates of vectors from the standard basis in the given basis.

7.2 Example

Let $\mathcal{A} = \{(2,3),(-1,2)\}$, and $\mathcal{B} = \{(2,1),(-1,2)\}$. Let us first find $M(id)_{\mathcal{A}}^{\mathcal{B}}$. To do this we calculate the coordinates of vectors from $\mathcal{A}$ with respect to $\mathcal{B}$. We solve two systems of equations (putting them into one matrix).

$$\left[\begin{array}{cc|cc} 2 & -1 & 2 & -1 \\ -1 & 2 & 3 & 2 \end{array}\right] \xrightarrow{w_1 \leftrightarrow w_2} \left[\begin{array}{cc|cc} -1 & 2 & 3 & 2 \\ 2 & -1 & 2 & -1 \end{array}\right] \xrightarrow{w_2 + 2w_1}$$

$$\left[\begin{array}{cc|cc} -1 & 2 & 3 & 2 \\ 0 & 3 & 8 & 3 \end{array}\right] \xrightarrow{w_1 \cdot (-1), w_2 \cdot 1/3} \left[\begin{array}{cc|cc} 1 & -2 & -3 & -2 \\ 0 & 1 & 8/3 & 1 \end{array}\right] \xrightarrow{w_1 + 2w_2} \left[\begin{array}{cc|cc} 1 & 0 & 7/3 & 0 \\ 0 & 1 & 8/3 & 1 \end{array}\right].$$

So the coordinates are $7/3, 8/3$ and $0, 1$. Thus,

$$M(id)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 7/3 & 0 \\ 8/3 & 1 \end{bmatrix}.$$

We get without any calculations that

$$M(id)_{\mathcal{A}}^{st} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}.$$

To calculate $M(id)_{st}^{\mathcal{B}}$, we have to find the coordinates of the vectors from the standard basis $(1,0) = 2/5(2,1) - 1/5(-1,2)$ and $(0,1) = 1/5(2,1) + 2/5(-1,2)$, so

$$M(id)_{st}^{\mathcal{B}} = \begin{bmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{bmatrix}.$$

7.3 Exemplary problems

Find $M(id)_{\mathcal{A}}^{\mathcal{B}}$, $M(id)_{\mathcal{A}}^{st}$ and $M(id)_{st}^{\mathcal{B}}$ for

- $\mathcal{A} = \{(1,2,3),(1,1,0),(-1,0,3)$, $\mathcal{B} = \{(0,1,1),(1,0,1),(0,1,0)\}$,
- $\mathcal{A} = \{(2,0,1,7),(1,0,1,0),(0,0,1,2),(0,0,-1,-1)\}$, $\mathcal{B} = \{(1,1,0,0),(0,1,1,0),(0,0,1,1),(1,0,0,1)\}$.

8 Changing bases of matrix of a linear transformation

8.1 Method

8.1.1 First method: calculate coordinates

Assume that we are given a formula defining ϕ, as above (or its matrix in the standard basis) and basis $\mathcal{A}$ and $\mathcal{B}$. E.g.: $\mathcal{A} = ((1,1,1),(1,0,-1),(1,0,0))$ and $\mathcal{B} = ((2,3),(-1,-1))$. We would like to calculate $M(\phi)_{\mathcal{A}}^{\mathcal{B}}$.

Notice that if we multiply this matrix from the right by (vertically) the first vector from the standard basis, then I will get simply the first column of this matrix. On the other hand the first vector v_1 from $\mathcal{A}$, has in this basis coordinates 1,0,0. Therefore the result of the multiplication are the coordinates of $\phi(v_1)$ in basis $\mathcal{B}$ and this is the first column of $M(\phi)_{\mathcal{A}}^{\mathcal{B}}$.

8.1.2 Metoda druga: mnożymy przez macierz zmiany bazy

We have a new tool to change basis of a matrix of a linear map. Because $id \circ \phi \circ id = \phi$, we have that:

$$M(\phi)_{\mathcal{A}}^{\mathcal{D}} = M(id \circ \phi \circ id)_{\mathcal{A}}^{\mathcal{D}} = M(id)_{\mathcal{C}}^{\mathcal{D}} \cdot M(\phi)_{\mathcal{B}}^{\mathcal{C}} \cdot M(id)_{\mathcal{A}}^{\mathcal{B}}.$$

In particular:

$$M(\phi)_{\mathcal{A}}^{\mathcal{B}} = M(id)_{st}^{\mathcal{B}} \cdot M(\phi)_{st}^{st} \cdot M(id)_{\mathcal{A}}^{st}$$

and in this way given a matrix of a linear map in standard basis we can calculate its matrix in basis from $\mathcal{A}$ to $\mathcal{B}$.

In the reverse direction we get

$$M(\phi)_{st}^{st} = M(id)_{\mathcal{B}}^{st} \cdot M(\phi)_{\mathcal{A}}^{\mathcal{B}} \cdot M(id)_{st}^{\mathcal{A}}.$$

If we want to change only one of the bases, we get respectively:

$$M(\phi)_{\mathcal{B}}^{\mathcal{D}} = M(id)_{\mathcal{C}}^{\mathcal{D}} \cdot M(\phi)_{\mathcal{B}}^{\mathcal{C}}.$$

and

$$M(\phi)_{\mathcal{A}}^{\mathcal{C}} = M(\phi)_{\mathcal{B}}^{\mathcal{C}} \cdot M(id)_{\mathcal{A}}^{\mathcal{B}}.$$

8.2 Example

Let, for example $\mathcal{A} = ((1,1,1),(1,0,-1),(1,0,0))$ and $\mathcal{B} = ((2,3),(-1,-1))$. We want to get $M(\phi)_{\mathcal{A}}^{\mathcal{B}}$. Let us use the first method.

So: $\phi((1,1,1)) = (1-1, 2+3-1) = (0,4)$ and we have to find the coordinates of this vector in $\mathcal{B}$: $(0,4) = 4(2,3) + 8(-1,-1)$, so the coordinates are 4,8 and this is the first column of the matrix we would like to calculate.

Let us find the second column. We do the same as before but with the second vector from basis $\mathcal{A}$. $\phi((1,0,-1)) = (1-(-1), 2-(-1)) = (2,3)$ has coordinates 1,0 in $\mathcal{B}$.

The third column: $\phi((1,0,0)) = (1,2)$ has in $\mathcal{B}$ coordinates $(1,2) = (2,3) + (-1,-1)$, so 1, 1 ¡p¿Therefore:

$$M(\phi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 4 & 1 & 1 \\ 8 & 0 & 1 \end{bmatrix}$$

In the second method we get the following.

$$M(id)_{\mathcal{A}}^{st} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

We calculate $M(id)_{st}^{\mathcal{B}}$. $(1,0) = -(2,3) - 3(-1,-1)$ and $(0,1) = (2,3) + 2(-1,-1)$. Therefore:

$$M(id)_{st}^{\mathcal{B}} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}.$$

We get:

$$M(\phi)_{\mathcal{A}}^{\mathcal{B}} = M(id)_{st}^{\mathcal{B}} \cdot M(\phi)_{st}^{st} \cdot M(id)_{\mathcal{A}}^{st},$$

so

$$M(\phi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 3 & 0 \\ 1 & 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 \\ 8 & 0 & 1 \end{bmatrix}.$$

8.3 Exemplary problems

- Let $\mathcal{A} = \{(1,1,0),(0,1,1),(1,0,1)\}$, $\mathcal{B} = \{(0,1),(1,-1)\}$ and $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be such that $\varphi((a,b,c) = (a+2b, -a-b-3c)$. Find $M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$.
- Let $\mathcal{A} = \{(1,1,0),(0,1,1),(1,0,1)\}$, $\mathcal{B} = \{(0,1),(1,-1)\}$ and $\psi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be such that $M(\psi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$. Find a formula for ψ.
- Let V be a linear space and $\varphi\colon V \to V$ be a linear transformation and $\mathcal{A},\mathcal{B}$ be bases of V, such that $M(\varphi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ and $M(id)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 4 & 3 & 6 \end{bmatrix}$. Find $M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$.
- Let $\mathcal{A} = \{(1,0,-1,0),(3,1,-2,0),(2,0,0,1),(-1,2,4,0)\}, \mathcal{B} = \{(2,1),(5,3)\}$,
  $\mathcal{C} = \{(1,0,1),(-2,1,-3),(-5,3,-9)\}$ and let $\phi\colon \mathbb{R}^4 \to \mathbb{R}^2, \varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ i $\psi\colon \mathbb{R}^2 \to \mathbb{R}^3$ be such that:
  - $\phi((a,b,c,d)) = (a-c+d, 2b-d)$,
  - $M(\varphi)_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}$,
  - $M(\psi)_{st}^{st} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix}$,

  Find: $M((\varphi + 2\psi) \circ \phi))_{\mathcal{A}}^{\mathcal{C}}$.

9 Calculating coordinates of values given a matrix of a linear transformation and coordinates of an argument

9.1 Method

Matrix of a linear transformation φ in bases $\mathcal{A}$ and $\mathcal{B}$ is designed is such a way that when multiplied by coordinates of a vector v with respect to $\mathcal{A}$, we get the coordinates of $\varphi(v)$ with respect to $\mathcal{B}$.

9.2 Example

Let $\varphi\colon V \to W$ and let $\mathcal{A}$ be a basis of V, $\mathcal{B}$ be a basis of W, and let $M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 1 & -1 & -1 & 4 \end{bmatrix}$. Assume that v with respect to $\mathcal{A}$ has coordinates $1, 1, 2, 0$. Let us find the coordinates of $\varphi(v)$ with respect to $\mathcal{B}$:

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 1 & -1 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix},$$

So the coordinates are $-1, 2$.

9.3 Exemplary problems

Let $\varphi\colon V \to W$ and let $\mathcal{A}$ be a basis V, $\mathcal{B}$ a basis of W. Assume that v with respect to $\mathcal{A}$ has coordinates $1, -1, 2$. Find the coordinates of $\varphi(v)$ with respect to $\mathcal{B}$, if

- $M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 3 & -2 & 0 \\ 1 & -1 & 5 \end{bmatrix}$,
- $M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 \\ 5 & -10 & 5 \end{bmatrix}$.

10 Calculating determinants of matrices 2×2 and 3×3

10.1 Method

Immediately from the definition we get

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Therefore,

$$\begin{vmatrix} a & b & c \\ k & l & m \\ x & y & z \end{vmatrix} = a \begin{vmatrix} l & m \\ y & z \end{vmatrix} - b \begin{vmatrix} k & m \\ x & z \end{vmatrix} + c \begin{vmatrix} k & l \\ x & y \end{vmatrix} =$$

$$= a(lz - my) - b(kz - mx) + c(ky - lx) = alz + bmx + cky - amy - bkz - clx.$$

10.2 Examples

$$\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 1 \cdot 5 - 3 \cdot 2 = -1,$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ 0 & 1 & 0 \end{vmatrix} = 1 \cdot (-2) \cdot 0 + 2 \cdot (-1) \cdot 0 + 3 \cdot (-3) \cdot 1 - 1 \cdot (-1) \cdot 1 - 2 \cdot (-3) \cdot 0 - 3 \cdot (-2) \cdot 0 = 0 + 0 - 9 + 1 - 0 - 0 = -8.$$

10.3 Exemplary problems

Calculate determinants of the following matrices:

$$\begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 3 \\ 3 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

11 Calculating determinant using Laplace extension

11.1 Method

The determinant of a matrix makes sense for square matrices only and is defined recursively:

- $\det[a] = a$
- $$\det \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix} =$$
$$= a_{1,1} \det A_{1,1} - a_{1,2} \det A_{1,2} + a_{1,3} \det A_{1,3} - \ldots \pm a_{1,n} \det A_{1,m},$$

where $A_{i,j}$ is matrix A with i-th row and j-th column crossed out. So (the determinant is denoted by det or by using absolute value style brackets around a matrix).

The above definition is only a special case of a more general fact called Laplace expansion. Instead of using the first row we can use any row or column (choose always the one with most zeros). So:

$$\det \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix} =$$

$$= (-1)^i \left(a_{i,1} \det A_{i,1} - a_{i,2} \det A_{i,2} + a_{i,3} \det A_{i,3} - \ldots \pm a_{i,n} \det A_{i,m}\right),$$

for any row w_i. Analogical fact is true for any column.

11.2 Example

Using the definition:

$$\begin{vmatrix} 1 & 0 & 2 & 0 \\ 2 & 3 & 0 & -1 \\ 3 & -1 & -1 & 0 \\ 0 & 1 & -1 & -2 \end{vmatrix} =$$

$$= 1 \begin{vmatrix} 3 & 0 & -1 \\ -1 & -1 & 0 \\ 1 & -1 & -2 \end{vmatrix} - 0 \begin{vmatrix} 2 & 0 & -1 \\ 3 & -1 & 0 \\ 0 & -1 & -2 \end{vmatrix} + 2 \begin{vmatrix} 2 & 3 & -1 \\ 3 & -1 & 0 \\ 0 & 1 & -2 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 & 0 \\ 3 & -1 & -1 \\ 0 & 1 & -1 \end{vmatrix} =$$

$$= 1 \cdot 0 - 0 + 2 \cdot 19 - 0 = 38$$

Using the Laplace's extension it is most convenient to use the third column:

$$\begin{vmatrix} 1 & 1 & 0 & -1 \\ 2 & 0 & -1 & -1 \\ 3 & -1 & 0 & 1 \\ 0 & 1 & 0 & -2 \end{vmatrix} = (-1)^3 \cdot (-1) \cdot \begin{vmatrix} 1 & 1 & -1 \\ 3 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix} = 10$$

11.3 Exemplary problems

Calculate the determinants of:

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & 3 & 0 & 1 \\ 4 & 1 & -1 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -2 & 0 \\ 2 & 0 & 0 & 1 \\ 4 & 1 & -1 & 1 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

12 Calculating determinant by transforming a matrix into triangular form

12.1 Method

Notice first that from the Laplace expansion we easily get that if a matrix has a row of zeros (or column) its determinant equals zero.

Consider now different operations on rows of a matrix, which we use to calculate a „stair-like" form of a matrix. Using Laplace expansion we can prove that **swapping two rows multiplies the determinant by** -1 – indeed calculating the determinant using the first column we see that the signs in the sum may change, but also the rows in the minor matrices get swapped.

Immediately we can notice that **multiplying a row by a number multiplies also the determinant by this number** – you can see it easily calculating Laplace expansion using this row. ¡p¿Therefore multiplying whole matrix by a number multiplies the determinant by this number many times, precisely:

$$\det(aA) = a^n \det A,$$

where A is a matrix of size $n \times n$.

Notice also, that the determinant of a matrix with two identical rows equals zero, because swapping those rows does not change the matrix but multiplies the determinant by -1, so $\det A = -\det A$, therefore $\det A = 0$. So because of the row multiplication rule, if two rows in a matrix are linearly dependent, then its determinant equals 0. ¡p¿Also the Laplace expansion implies that if matrices A, B, C differ only by i-th row in the way that this row in matrix C is a sum of i-th rows in matrices B and C, then the determinant of C is the sum of determinants of A and B, e.g.:

$$\begin{vmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 3 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & 3 & 3 \end{vmatrix}.$$

But it can be easily seen that in general $\det(A + B) \neq \det A + \det B$!

Finally, consider the most important operation of adding to a row another row multiplied by a number. Then we actually deal with the situation described above. The resulting matrix is matrix C, which differs from A and B only by the row we sum to. Matrix A is the original matrix and matrix C is matrix A, in which we substitute the row we sum to with the row we are summing multiplied by a number. Therefore $\det A = \det B + \det C$, but C has two linearly dependent rows, so $\det C = 0$ and $\det B = \det A$. Therefore **the operation of adding a row multiplied by a number to another row does not change the determinant of a matrix**.

Finally, the matrix multiplication. All the above operations can be written as multiplication by a special matrix. E.g. swapping of 2-nd and 3-rd rows in a matrix of size 3×3, is actually the following:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot$$

multiplication of the 3-rd row by scalar 4, is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \cdot$$

adding to the third row the second multiplied by 2 is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot$$

It is relatively easy to see that even in the general case the matrix, which changes rows has determinant (-1), the matrix multiplying a row by scalar a, has determinant a, and matrix of adding to a row another row multiplied by a scalar has determinant 1. Therefore multiplying by those matrices (called the elementary matrices) multiplies the determinant of a matrix by their determinants. Moreover, every matrix can be created by multiplying elementary matrix, which gives the following important conclusion:

$$\det(A \cdot B) = \det A \cdot \det B$$

If you look closely enough you will see that the Laplace expansion also implies that the determinant of a matrix in an echelon form (usually called triangular for square matrices) equals the product of elements on the diagonal of the matrix, so e.g.:

$$\begin{vmatrix} -1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{vmatrix} = (-1) \cdot 1 \cdot 3 = -3.$$

Because we know how the elementary operations change, to calculate the determinant of a matrix we can calculate a triangular form, calculate its determinant and recreate the determinant of the original matrix. This method is especially useful for large matrices.

12.2 Example

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 6 & 2 & 0 & 2 \\ 3 & 9 & 1 & 1 & -1 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{w_2 - 2w_1, w_3 - 3w_1, w_4 - w_1, w_5 - w_1}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 2 & 2 & -6 & 0 \\ 0 & 3 & 1 & -8 & -4 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{1}{2}}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 1 & 1 & -3 & 0 \\ 0 & 3 & 1 & -8 & -4 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} \xrightarrow{w_3 - 3w_2, w_4 \leftrightarrow w_5} \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & -2 & 1 & -4 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Therefore, the determinant of the last matrix is $1 \cdot 1 \cdot (-2) \cdot (-2) \cdot 3 = 12$. On our way we have swapped rows once and we have multiplied one row by $\frac{1}{2}$, therefore the determinant of the first matrix equals $\frac{12 \cdot (-1)}{\frac{1}{2}} = -24$.

12.3 Exemplary problems

Calculate the determinant transforming the matrix into triangular form

$$\begin{bmatrix} 6 & -1 & 2 & 0 & 1 \\ 6 & 3 & 0 & 1 & 1 \\ 9 & 3 & -3 & 6 & -3 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 1 & 1 & 0 & -1 \end{bmatrix}.$$

13 Calculating determinant of a matrix in a block form

13.1 Method

The above fact also implies how to calculate the determinant of a matrix which is in the block form: $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ with left bottom block of zeros. The determinant of such a matrix equals $\det A \cdot \det B$

13.2 Example

$$\begin{vmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 6 & 2 & 0 & 2 \\ 3 & 9 & 1 & 1 & -1 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 6 & 2 \\ 3 & 9 & 1 \end{vmatrix} \cdot \begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix}.$$

13.3 Exemplary problems

Calculate determinants of:

$$\begin{bmatrix} -1 & 3 & 0 & 2017 & 1 \\ 2 & 6 & 2 & 0 & 2018 \\ 1 & 0 & 1 & 2016 & -1 \\ 0 & 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 1 & 2017 & 0 & 2017 & 0 \\ 2 & 6 & 2 & 0 & 20 & 2017 & 0 & 2017 & \\ 3 & 9 & 1 & 1 & -1 & 2017 & 0 & 2017 & 0 \\ 1 & 2 & 0 & 3 & 4 & 0 & 2017 & 0 & 2017 \\ 1 & 2 & 0 & 1 & 1 & 2017 & 0 & 2017 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 \end{bmatrix}$$

14 Calculating determinant using known facts

14.1 Method

Let me once more remind all the facts we can use to calculate the determinant of a matrix.

Notice first that from the Laplace expansion we easily get that if a matrix has a row of zeros (or column) its determinant equals zero.

Consider now different operations on rows of a matrix, which we use to calculate a „stair-like" form of a matrix. Using Laplace expansion we can prove that **swapping two rows multiplies the determinant by** -1 – indeed calculating the determinant using the first column we see that the signs in the sum may change, but also the rows in the minor matrices get swapped.

Immediately we can notice that **multiplying a row by a number multiplies also the determinant by this number** – you can see it easily calculating Laplace expansion using this row. ¡p¿Therefore multiplying whole matrix by a number multiplies the determinant by this number many times, precisely:

$$\det(aA) = a^n \det A,$$

where A is a matrix of size $n \times n$.

Notice also, that the determinant of a matrix with two identical rows equals zero, because swapping those rows does not change the matrix but multiplies the determinant by -1, so $\det A = -\det A$, therefore $\det A = 0$. So because of the row multiplication rule, if two rows in a matrix are linearly dependent, then its determinant equals 0. ¡p¿Also the Laplace expansion implies that if matrices A, B, C differ only by i-th row in the way that this row in matrix C is a sum of i-th rows in matrices B and C, then the determinant of C is the sum of determinants of A and B, e.g.:

$$\begin{vmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 3 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & 3 & 3 \end{vmatrix}.$$

But it can be easily seen that in general $\det(A+B) \neq \det A + \det B$!

Finally, consider the most important operation of adding to a row another row multiplied by a number. Then we actually deal with the situation described above. The resulting matrix is matrix C, which differs from A and B only by the row we sum to. Matrix A is the original matrix and matrix C is matrix A, in which we substitute the row we sum to with the row we are summing multiplied by a number. Therefore $\det A = \det B + \det C$, but C has two linearly dependent rows, so $\det C = 0$ and $\det B = \det A$. Therefore **the operation of adding a row multiplied by a number to another row does not change the determinant of a matrix**.

Finally, the matrix multiplication. All the above operations can be written as multiplication by a special matrix. E.g. swapping of 2-nd and 3-rd rows in a matrix of size 3×3, is actually the following:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot$$

multiplication of the 3-rd row by scalar 4, is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \cdot$$

adding to the third row the second multiplied by 2 is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot$$

It is relatively easy to see that even in the general case the matrix, which changes rows has determinant (-1), the matrix multiplying a row by scalar a, has determinant a, and matrix of adding to a row another row multiplied by a scalar has determinant 1. Therefore multiplying by those matrices (called the elementary matrices) multiplies the determinant of a matrix by their determinants. Moreover, every matrix can be created by multiplying elementary matrix, which gives the following important conclusion:

$$\det(A \cdot B) = \det A \cdot \det B$$

14.2 Example

Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 6 & 2 & 0 & 2 \\ 3 & 9 & 1 & 1 & -1 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 1 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 6 & 2 & 0 & 2 \\ 3 & 9 & 1 & 1 & -1 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 3 & 1 \end{bmatrix}.$$

We have calculated before that $\det A = -24$. But, $\det B = 0$, because it has two equal rows.
Therefore, $\det(A/2) = -24/2^5 = -24/32 = 3/4$.
And also $\det(A \cdot B) = (-24) \cdot 0 = 0$.

14.3 Exemplary problems

For the matrices defined above calculate

$$\det(2A),$$

$$\det(A^2).$$

15 Calculating inverse matrix

15.1 Method

A matrix B is inverse to matrix A, if $A \cdot B = I$, where I is the identity matrix (the matrix with ones on the diagonal and zeros everywhere else). The inverse matrix is denoted as A^{-1}. Since $\det A \cdot B = \det A \cdot \det B$ and $\det I = 1$, we see that $\det A^{-1} = \frac{1}{\det A}$. This implies that only matrices with non-zero determinants can have their inverses. Therefore we call such matrices invertible.

How to calculate the inverse of a given matrix? We have mentioned recently that the operations on rows of a matrix leading to the reduced "stair-like" for is actually multiplication by a matrix. Imagine that we transform the matrix $[A|I]$ consisting of matrix A along with the identity matrix into the reduced "stair-like" form. Since A is a square matrix with non-zero determinant, we will get identity matrix on the left side: $[I|B]$. But notice that if C is the matrix of rows operations, then $C \cdot [A|I] = [I|B]$. Therefore $C \cdot A = I$ and $C \cdot I = B$. The first equation implies that $C = A^{-1}$. The second that $B = C = A^{-1}$. So we get the inverse matrix on the right after those operations!

15.2 Example

Let us calculate the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -1 & -2 & 0 \end{bmatrix}$$

So:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 0 & 1 & 0 \\ -1 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[w_2 - 2w_1, w_3 + w_1]{} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow[w_1 - w_3]{}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow[w_1 - 2w_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

And therefore:

$$A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

15.3 Exemplary problems

If a matrix is invertible, calculate its inverse

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 6 & 2 & 0 & 2 \\ 3 & 9 & 1 & 1 & -1 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 6 & 2 & 0 & 2 \\ 3 & 9 & 1 & 1 & -1 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 1 & 1 \end{bmatrix}$$

16 Calculating one coefficient of an inverse matrix

16.1 Method

If you do not need the whole matrix but some elements only, the following method seems useful. It uses the adjugate matrix to the given one. The adjugate matrix is a matrix in which in j-th row and i-th column we have the determinant of matrix $A_{i,j}$ (matrix A without i-th row and j-th column, there is no mistake there, a transposition plays a role here) multiplied by $(-1)^{i+j}$. The following equation holds:

$$A^{-1} = \frac{A^D}{\det A}.$$

16.2 Example

Let, as before

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -1 & -2 & 0 \end{bmatrix}.$$

If we want to calculate the value in the second row and first column of A^{-1} from the previous example we cross out the second column an the first row of A and calculate the determinants, and get:

$$(-1)^{2+1} \frac{\begin{vmatrix} 2 & 2 \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -1 & -2 & 0 \end{vmatrix}} = (-1)\frac{2}{1} = -2$$

which agrees with the result obtained by the first method!

16.3 Exemplary problems

Using this method find also

- entry in the second row and third column,
- entry in the third row and second column

of matrix A^{-1}, for A from the above example.

17 Solving a system of linear equations using Cramer's rule

17.1 Method

Given a system of n equations with n variables we may try to solve it with Cramer's rule. Let A be the matrix of this system without the column of free coefficients. Let A_i be the matrix A, in which instead of i-th column we put the column of free coefficients. Then:

- if $\det A \neq 0$, the system has exactly one solution. The solution is given by the following formula: $x_i = \frac{\det A_i}{\det A}$,
- if $\det A = 0$, and at least one of $\det A_i$ is not equal to 0, the system has no solutions,
- if $\det A = 0$ and for every i, $\det A_i = 0$, there can be zero or infinitely many solutions – Cramer's method does not give any precise answer.

17.2 Example

Let us solve the following system of equations:

$$\begin{cases} x + 2y + z = 1 \\ 2x + 5y + 2z = -1 \\ -x - 2y = 0 \end{cases}$$

Therefore:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -1 & -2 & 0 \end{bmatrix}$$

Since $\det A = 1$, this system has exactly one solution. To determine it we calculate the other determinants:

$$\det A_1 = \begin{vmatrix} 1 & 2 & 1 \\ -1 & 5 & 2 \\ 0 & -2 & 0 \end{vmatrix} = 6$$

$$\det A_2 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ -1 & 0 & 0 \end{vmatrix} = -3$$

$$\det A_3 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & -1 \\ -1 & -2 & 0 \end{vmatrix} = 1$$

And so $x = \frac{6}{1} = 6$, $y = \frac{-3}{1} = -3$, $z = \frac{1}{1} = 1$.

17.3 Exemplary problems

Check the following systems using Cramer's rules. If a system has exactly one solution, calculate it using Cramer's rules.

$$\begin{cases} x + 2y + z = 1 \\ 2x + 5y + 2z = -1 \\ 3x + 7y + 3z = 1 \end{cases},$$

$$\begin{cases} 2x + 5y - 10z = -1 \\ x + 3y - 2z = 0 \\ -x + 5z = 33 \end{cases}.$$