Linear algebra , WNE, 2017/2018
Exemplary problems for the 2nd test

28 November 2017

Problem 1.

Let $\mathcal{A} = \{(1,0,-1,0),(3,1,-2,0),(2,0,0,1),(-1,2,4,0)\}, \mathcal{B} = \{(2,1),(5,3)\}$,
$\mathcal{C} = \{(1,0,1),(-2,1,-3),(-5,3,-9)\}$ and let $\phi\colon \mathbb{R}^4 \to \mathbb{R}^2, \varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ i $\psi\colon \mathbb{R}^2 \to \mathbb{R}^3$ be such that:

- $\phi((a,b,c,d)) = (a-c+d, 2b-d)$,
- $M(\varphi)_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}$,
- $M(\psi)_{\text{st}}^{\text{st}} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix}$,

Find:

- $M((\varphi + 2\psi)\circ\phi)_{\mathcal{A}}^{\mathcal{C}}$,
- the coordinates of $\varphi(\phi(\alpha)) + 2\psi(\phi(\alpha))$ with respect to $\mathcal{C}$, if the coordinates of vector α with respect to $\mathcal{A}$ are $1,-1,0,1$.
- the coefficient in the second row and third column in $(M(\text{id})_{\mathcal{A}}^{\text{st}})^{-1}$.

Solution: $M((\varphi+2\psi)\circ\phi)_{\mathcal{A}}^{\mathcal{C}} = M(\varphi+2\psi)_{\text{st}}^{\mathcal{C}} \cdot M(\phi)_{\mathcal{A}}^{\text{st}} = (M(\varphi)_{\text{st}}^{\mathcal{C}} + 2M(\psi)_{\text{st}}^{\mathcal{C}}) \cdot M(\phi)_{\mathcal{A}}^{\text{st}} = (M(\varphi)_{\mathcal{B}}^{\mathcal{C}} \cdot M(\text{id})_{\text{st}}^{\mathcal{B}} + 2M(\text{id})_{\text{st}}^{\mathcal{C}} \cdot M(\psi)_{\text{st}}^{\text{st}}) \cdot M(\phi)_{\text{st}}^{\text{st}} \cdot M(\text{id})_{\mathcal{A}}^{\text{st}}$.

We need the coordinates of the vectors from the standard basis with respect to $\mathcal{B}$ i $\mathcal{C}$. We get: $(1,0) = (3,-1)_{\mathcal{B}}, (0,1) = (-5,2)_{\mathcal{B}}$ and $(1,0,0) = (0,-3,1)_{\mathcal{C}}, (0,1,0) = (3,4,-1)_{\mathcal{C}}, (0,0,1) = (1,3,-1)_{\mathcal{C}}$, so

$$M(\text{id})_{\text{st}}^{\mathcal{B}} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix},$$

$$M(\text{id})_{\text{st}}^{\mathcal{C}} = \begin{bmatrix} 0 & 3 & 1 \\ -3 & 4 & 3 \\ 1 & -1 & -1 \end{bmatrix}.$$

Obviously,

$$M(\text{id})_{\mathcal{A}}^{\text{st}} = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$M(\phi)_{\text{st}}^{\text{st}} = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 0 & -1 \end{bmatrix}.$$

We do the calculations:

$$M((\varphi+2\psi)\circ\phi)_{\mathcal{A}}^{\mathcal{C}} =$$

$$= \left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} + 2 \begin{bmatrix} 0 & 3 & 1 \\ -3 & 4 & 3 \\ 1 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix} \right) \cdot$$

$$\cdot \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix} =$$

$$= \left(\begin{bmatrix} 3 & -5 \\ -4 & 7 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} 12 & -2 \\ 10 & 0 \\ -2 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 2 & 5 & 3 & -5 \\ 0 & 2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 15 & -7 \\ 6 & 7 \\ -4 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 & 3 & -5 \\ 0 & 2 & -1 & 4 \end{bmatrix} =$$

$$= \begin{bmatrix} 30 & 61 & 52 & -103 \\ 12 & 44 & 11 & -2 \\ -8 & -12 & -16 & 36 \end{bmatrix}.$$

The coordinates which we are looking for are

$$\begin{bmatrix} 30 & 61 & 52 & -103 \\ 12 & 44 & 11 & -2 \\ -8 & -12 & -16 & 36 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -134 \\ -34 \\ 40 \end{bmatrix}.$$

Let $A = M(\mathrm{id})^{\mathrm{st}}_{\mathcal{A}} = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Then the element in in the second row and third column of A^{-1} is $(-1)^{2+3} \frac{\det A_{2,3}}{\det A}$. We calculate those determinants:

$$\det A_{2,3} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2,$$

$$\det A = (-1)^{3+4} \begin{vmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ -1 & -2 & 4 \end{vmatrix} = (-1)(4-6-1+4) = -1$$

So this element is $(-1)\frac{-2}{-1} = -2$.

Problem 2.

Let V be a linear space and $\varphi\colon V \to V$ be a linear transformation and $\mathcal{A}, \mathcal{B}$ be bases of V, such that $M(\varphi)^{\mathcal{A}}_{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ and $M(id)^{\mathcal{B}}_{\mathcal{A}} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 4 & 3 & 6 \end{bmatrix}$. Find

- $M(\varphi)^{\mathcal{B}}_{\mathcal{A}}$,
- $M(\varphi)^{\mathcal{B}}_{\mathcal{B}}$ (hint: $(M(id)^{\mathcal{B}}_{\mathcal{A}})^{-1} = M(id)^{\mathcal{A}}_{\mathcal{B}}$),
- coordinates of $\varphi(\alpha)$ with respect to $\mathcal{B}$, if the coordinates of vector α with respect to $\mathcal{A}$ are $1, -2, 2$.

Solution:

$$M(\varphi)^{\mathcal{B}}_{\mathcal{A}} = M(\mathrm{id})^{\mathcal{B}}_{\mathcal{A}} \cdot M(\varphi)^{\mathcal{A}}_{\mathcal{A}} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 4 & 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 11 & 3 \\ 1 & 5 & 1 \\ 7 & 21 & 2 \end{bmatrix}.$$

$$M(\varphi)^{\mathcal{B}}_{\mathcal{B}} = M(\varphi)^{\mathcal{B}}_{\mathcal{A}} \cdot M(\mathrm{id})^{\mathcal{A}}_{\mathcal{B}} = M(\varphi)^{\mathcal{B}}_{\mathcal{A}} \cdot (M(\mathrm{id})^{\mathcal{B}}_{\mathcal{A}})^{-1}.$$

We have to calculate the inverse matrix.

$$\begin{bmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 4 & 3 & 6 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_2}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 4 & 3 & 6 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{w_2 - 2w_1, w_3 - 4w_1} \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 & -4 & 1 \end{bmatrix} \xrightarrow{w_3 + w_2}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & -6 & 1 \end{bmatrix} \xrightarrow{w_1 - w_3, w_2 + w_3} \begin{bmatrix} 1 & 1 & 0 & -1 & 7 & -1 \\ 0 & 1 & 0 & 2 & -8 & 1 \\ 0 & 0 & 1 & 1 & -6 & 1 \end{bmatrix} \xrightarrow{w_1 - w_2}$$

$$\begin{bmatrix} 1 & 0 & 0 & -3 & 15 & -2 \\ 0 & 1 & 0 & 2 & -8 & 1 \\ 0 & 0 & 1 & 1 & -6 & 1 \end{bmatrix},$$

Hence,

$$M(\varphi)_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 0 & 11 & 3 \\ 1 & 5 & 1 \\ 7 & 21 & 2 \end{bmatrix} \cdot \begin{bmatrix} -3 & 15 & -2 \\ 2 & -8 & 1 \\ 1 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 25 & -106 & 14 \\ 8 & -31 & 4 \\ 23 & -75 & 9 \end{bmatrix}.$$

The coordinates we are looking for are

$$M(\varphi)_{\mathcal{A}}^{\mathcal{B}} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -16 \\ -7 \\ -31 \end{bmatrix}.$$

Problem 3.

Using Cramer's rules determine for which $s, t \in \mathbb{R}$,

$$\begin{cases} 3x + sy + z = 5 \\ 8x + 5y + 8z = t \\ 2x + y + 6z = 6 \end{cases}$$

has exactly one solution. Using Cramer's rules find the solution for $t = s = 0$ and prove that if $t \neq 16$ it is inconsistent except for the case when it has exactly one solution.

Rozwiązanie: $\det A = 90 + 16s + 8 - 10 - 24 - 48s = -32s + 64$, so for $s \neq 2$ the system has exactly one solution.

If $s = t = 0$, then $\det A = 64, \det A_1 = 150 - 30 - 40 = 80, \det A_2 = 80 + 48 - 240 - 144 = -256$ and $\det A_3 = 90 + 40 - 50 = 80$, so $x = \frac{80}{64} = \frac{5}{4}, y = \frac{-256}{64} = -4$ and $z = \frac{5}{4}$.

Hence, assume that $s = 2$, and let us check that if $t \neq 16$ the system is inconsistent. So we have to check that at least one of the determinants in not equal to zero: $\det A_1 = 150 + 96 + t - 30 - 40 - 12t = 176 - 11t, \det A_2 = 18t + 80 + 48 - 2t - 240 - 144 = 16t - 256$ and $\det A_3 = 90 + 4t + 40 - 50 - 3t - 96 = t - 16$. So indeed for $t \neq 16$, actually all of them are non-zero.

Problem 4.

Let

$$A = \begin{bmatrix} s & 1 & 5 & 0 & 13 & -23 & 0 & -7 & -9 \\ 0 & -3 & 0 & 0 & -99 & 7 & -2 & 3 & 62 \\ -2 & 2 & s & 1 & 0 & 5 & 21 & -12 & 7 \\ 3 & 1 & 4 & -1 & 33 & -2 & 4 & 0 & -5 \\ 0 & 0 & 0 & 0 & -2 & 2 & 5 & -11 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & -9 & 15 & s \\ 0 & 0 & 0 & 0 & 3 & 4 & 17 & 13 & -2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 10 & 1 & 0 \end{bmatrix}.$$

For which real numbers $s \in \mathbb{R}$ matrix A is invertible? Calculate $\det(A^2 \cdot A^T \cdot (2 \cdot A)^{-3})$ assuming that A is invertible.

Solution: $\det(A^2 \cdot A^T \cdot (2 \cdot A)^{-3}) = \det A^2 \cdot \det A^T \cdot \det(2 \cdot A)^{-3} = (\det A)^2 \cdot \det A \cdot (\det 2 \cdot A)^{-3} = (\det A)^3 \cdot (2^9 \cdot \det A)^{-3} = (\det A)^3 \cdot 2^{-27} \cdot (\det A)^{-3} = \frac{1}{2^{27}}$.

We have to check for which s matrix A is invertible. In other words, for which s its determinant is not equal to zero. And since it is a matrix in a block form, we have to calculate the determinants of two matrices. The determinant of the whole matrix is the product of those two products. To calculate the first determinant we use Laplace'a extension:

$$\begin{vmatrix} s & 1 & 5 & 0 \\ 0 & -3 & 0 & 0 \\ -2 & 2 & s & 1 \\ 3 & 1 & 4 & -1 \end{vmatrix} = -3 \cdot (-1)^{2+2} \begin{vmatrix} s & 5 & 0 \\ -2 & s & 1 \\ 3 & 4 & -1 \end{vmatrix} = -3(-s^2 + 15 - 10 - 4s) = 3(s^2 + 4s - 5).$$

$\Delta = 16 + 20 = 36$, so the roots are $s_1 = -5, s_2 = 1$ and for such s the determinant is zero.

To calculate the determinant of the second matrix we transform it into triangular form.

$$\begin{bmatrix} -2 & 2 & 5 & -11 & -1 \\ 1 & 0 & -2 & 2 & 0 \\ 0 & -6 & -9 & 15 & s \\ 3 & 4 & 17 & 13 & -2 \\ 0 & 2 & 10 & 1 & 0 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_2} \begin{bmatrix} 1 & 0 & -2 & 2 & 0 \\ -2 & 2 & 5 & -11 & -1 \\ 0 & -6 & -9 & 15 & s \\ 3 & 4 & 17 & 13 & -2 \\ 0 & 2 & 10 & 1 & 0 \end{bmatrix} \xrightarrow{w_2 + 2w_1, w_4 - 3w_1}$$

$$\begin{bmatrix} 1 & 0 & -2 & 2 & 0 \\ 0 & 2 & 1 & -7 & -1 \\ 0 & -6 & -9 & 15 & s \\ 0 & 4 & 23 & 7 & -2 \\ 0 & 2 & 10 & 1 & 0 \end{bmatrix} \xrightarrow{w_3 + 3w_2, w_4 - 2w_2, w_5 - w_2} \begin{bmatrix} 1 & 0 & -2 & 2 & 0 \\ 0 & 2 & 1 & -7 & -1 \\ 0 & 0 & -6 & -6 & s-3 \\ 0 & 0 & 21 & 21 & 0 \\ 0 & 0 & 9 & 8 & 1 \end{bmatrix} \xrightarrow{w_3 \leftrightarrow w_4}$$

$$\begin{bmatrix} 1 & 0 & -2 & 2 & 0 \\ 0 & 2 & 1 & -7 & -1 \\ 0 & 0 & 21 & 21 & 0 \\ 0 & 0 & -6 & -6 & s-3 \\ 0 & 0 & 9 & 8 & 1 \end{bmatrix} \xrightarrow{w_3 \cdot \frac{1}{7}} \begin{bmatrix} 1 & 0 & -2 & 2 & 0 \\ 0 & 2 & 1 & -7 & -1 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & -6 & -6 & s-3 \\ 0 & 0 & 9 & 8 & 1 \end{bmatrix} \xrightarrow{w_4 + 2w_3, w_5 - 3w_3}$$

$$\begin{bmatrix} 1 & 0 & -2 & 2 & 0 \\ 0 & 2 & 1 & -7 & -1 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & s-3 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{w_4 \leftrightarrow w_5} \begin{bmatrix} 1 & 0 & -2 & 2 & 0 \\ 0 & 2 & 1 & -7 & -1 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & s-3 \end{bmatrix}$$

So the determinant of the last matrix is $1 \cdot 2 \cdot 3 \cdot (-1) \cdot (s-3) = -6(s-3)$. And we have swapped rows 3 three times and we have multiplied a row by $\frac{1}{7}$, so the determinant of the original matrix is $(-1)^3 \cdot 7 \cdot (-6)(s-3) = 42(s-3)$, which gives zero for $s = 3$.

So, finally, A is invertible for $s \in (-\infty, -5) \cup (-5, 1) \cup (1, 3) \cup (3, \infty)$.