Linear algebra , WNE, 2018/2019
meeting 18. – solutions

4 December 2018

1. For the following endomorphism $\varphi\colon V \to V$ check whether there exists a basis $\mathcal{A}$ of V which consists of eigenvectors of φ. If so, find $M(\varphi)_{\mathcal{A}}^{\mathcal{A}}$.
$$V = \mathbb{R}^2, \varphi((a,b)) = (a-b, a+3b)$$
We get
$$w(\lambda) = (1-\lambda)(3-\lambda)+1 = \lambda^2-4\lambda+4,$$
So we have one eigenvalue: 2. Eigenspace for eigenvalue 2 is spanned by $(1,-1)$, so it has only one-dimensional, and a basis of V consisting eigenvectors does not exist.

2. Check whether matrices $A_1 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$ are diagonalizable. If so, find matrices C_i such that $C_i^{-1}A_iC_i$ is diagonal for $i=1,2$.
We get $w_1(\lambda) = (1-\lambda)(3-\lambda)+1 = \lambda^2-4\lambda+4$. So we have only one eigenvalue 2, and its eigenspace is spanned by $(1,1)$, so we cannot get a basis which consists of eigenvectors and hence A_1 is not diagonalizable.
For the second matrix we get $w_2(\lambda) = (5-\lambda)(-1-\lambda)+9 = \lambda^2-4\lambda+4$. Similarly we also have only one eigenvalue 2, and similarly the eigenspace is spanned by $(1,1)$, so again there is no basis consisting of eigenvectors, and the matrix is not diagonalizable.

3. Check whether the following matrix A are diagonalizable. If so, find a matrix C such that $C^{-1}AC$ is diagonal.
$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{bmatrix}$$
   - $$w(\lambda) = ((1-\lambda)(-2-\lambda)-4)(-3-\lambda) = (\lambda^2+\lambda-6)(-3-\lambda) = (2-\lambda)(-3-\lambda)^2$$
   So the eigenvalues are $-3, 2$
   $$V_{(-3)}: \begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
   We get basis $\{(1,-2,0),(0,0,1)\}$.
   $$V_{(2)}: \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$
   We get basis $\{(2,1,0)\}$.
   So we get the basis of the whole space which consists of eigenvectors $\mathcal{A} = \{(1,-2,0),(0,0,1),(2,1,0)\}$, so the matrix is diagonalizable and its diagonal form is $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, and
   $$C = M(\mathrm{id})_{\mathcal{A}}^{\mathrm{st}} = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

- $$w(\lambda) = -\lambda(4-\lambda)(2-\lambda) + 8 - 4\lambda = -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda-2)^3$$
  So the only eigenvalue is 2, and
  $$V_{(2)} : \begin{bmatrix} -2 & 1 & 0 \\ -4 & -2 & 0 \\ -2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
  We get basis $\{(2,1,0),(0,0,1)\}$, so there is no basis of eigenvectors of the whole space, and the matrix is not diagonalizable.

4. For the following matrix A calculate A^{2019}.
$$\begin{bmatrix} 1 & -8 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

- $$w(\lambda) = (1-\lambda)(7-\lambda) + 8 = \lambda^2 - 8\lambda + 15 = (\lambda-3)(\lambda-5),$$
  we get eigenvalue 3 and 5.
  $$V_{(3)} : \begin{bmatrix} -2 & -8 \\ 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix},$$
  so the basis is $\{(-4,1)\}$,
  $$V_{(5)} : \begin{bmatrix} -4 & -8 \\ 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$
  so the basis is $\{(-2,1)\}$, so we get a basis of eigenvectors $\mathcal{A} = \{(-4,1)(-2,1)\}$, so
  $$A^{2019} = \begin{bmatrix} -4 & -2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3^{2019} & 0 \\ 0 & 5^{2019} \end{bmatrix} \cdot (-1/2) \cdot \begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix} =$$
  $$= \begin{bmatrix} -4 \cdot 3^{2019} & 2 \cdot 5^{2019} \\ 3^{2019} & 5^{2019} \end{bmatrix} \cdot (-1/2) \cdot \begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix} =$$
  $$= 1/2 \begin{bmatrix} 4 \cdot 3^{2019} + 2 \cdot 5^{2019} & 8 \cdot 3^{2019} + 8 \cdot 5^{2017} \\ -3^{2017} + 5^{2019} & 2 \cdot -3^{2019} + 4 \cdot 5^{2019} \end{bmatrix}.$$

- $$w(\lambda) = (1-\lambda)((2-\lambda)^2 - 1) = (1-\lambda)(\lambda^2 - 4\lambda + 3) = -(\lambda-1)^2(\lambda-3)$$
  we get eigenvalues 1 and 3.
  $$V_{(1)} : \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
  and so we get basis $\{(1,0,0),(0,1,-1)\}$.
  $$V_{(3)} : \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$
  so the basis is $\{(0,1,1)\}$. We get basis of eigenvectors: $\mathcal{A} = \{(1,0,0),(0,1,-1),(0,1,1)\}$.
  We have to calculate $M(\mathrm{id})^{\mathcal{A}}_{st}$. $(1,0,0) = (1,0,0)_{\mathcal{A}}, (0,1,0) = (0,\frac{1}{2},\frac{1}{2})_{\mathcal{A}}$ and $(0,0,1) = (0,-\frac{1}{2},\frac{1}{2})_{\mathcal{A}}$, so $M(\mathrm{id})^{\mathcal{A}}_{\mathrm{st}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Hence,
  $$A^{2019} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^{2019} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3^{2019} \\ 0 & 1 & 3^{2019} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} =$$
  $$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(1-3^{2019}) & \frac{1}{2}(-1+3^{2019}) \\ 0 & \frac{1}{2}(1+3^{2019}) & \frac{1}{2}(-1+3^{2019}) \end{bmatrix}.$$