Linear algebra, WNE, 2018/2019
meeting 18. – solutions

4 December 2018

Group 8:00

1. Check whether the following matrix A is diagonalizable. If so, find a matrix C such that $C^{-1}AC$ is diagonal.

$$\begin{bmatrix} -6 & 2 & 2 \\ 0 & 2 & 4 \\ 0 & 4 & -4 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) We calculate the characteristic polynomial

$$w(\lambda) = \begin{vmatrix} -6-\lambda & 2 & 2 \\ 0 & 2-\lambda & 4 \\ 0 & 4 & -4-\lambda \end{vmatrix} = (-6-\lambda)(\lambda^2+2\lambda-24) = -(6+\lambda)^2(\lambda-4),$$

So the eigenvalues are -6 and 4. The eigenspaces are

$$V_{(-6)}: \begin{bmatrix} 0 & 2 & 2 \\ 0 & 8 & 4 \\ 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we get a basis of $V_{(-6)}$: $\{(1,0,0)\}$.

$$V_{(4)}: \begin{bmatrix} -10 & 2 & 2 \\ 0 & -2 & 4 \\ 0 & 4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{3}{5} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

So we get a basis of $V_{(4)}$: $\{(\frac{3}{5},-2,1)\}$. Basis of those spaces together do not give a basis of the whole space $\mathbb{R}^3$, so the matrix is not diagonalizable.

(b) We calculate the characteristic polynomial

$$w(\lambda) = \begin{vmatrix} 2-\lambda & 1 & 3 \\ 1 & 2-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{vmatrix} = -\lambda^3+6\lambda^2-11a+6 = -(\lambda-1)(\lambda-2)(\lambda-3).$$

We get three eigenvalues $1,2,3$. We get the following bases of eigenspaces:

$$V_{(1)}: \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So we get a basis of $V_{(1)}$: $\{(-1,1,0)\}$.

$$V_{(2)}: \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

So we get a basis of $V_{(2)}$: $\{(-4,-3,1)\}$.

$$V_{(3)}: \begin{bmatrix} -1 & 1 & 3 \\ 1 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So we get a basis of $V_{(3)}$: $\{(1,1,0)\}$. Hence, we get basis of eigenvectors $\mathcal{A} = \{(-1,1,0),(-4,-3,1),(1,1,0)\}$. Thus,

$$C = M(\mathrm{id})^{st}_{\mathcal{A}} = \begin{bmatrix} -1 & -4 & 1 \\ 1 & -3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C^{-1}AC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

2. Calculate $\begin{bmatrix} 0 & 0 & 6 \\ 1 & 0 & -11 \\ 0 & 1 & 6 \end{bmatrix}^{2019}$.

We get the characteristic polynomial

$$w(\lambda) = \begin{vmatrix} -\lambda & 0 & 6 \\ 1 & -\lambda & -11 \\ 0 & 1 & 6-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = -(\lambda-1)(\lambda-2)(\lambda-3).$$

The eigenvalues are $1, 2, 3$. We study the eigenspaces:

$$V_{(1)}: \begin{bmatrix} -1 & 0 & 6 \\ 1 & -1 & -11 \\ 0 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

We get a basis of $V_{(1)}$: $\{(6,-5,1)\}$.

$$V_{(2)}: \begin{bmatrix} -2 & 0 & 6 \\ 1 & -2 & -11 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

We get a basis of $V_{(2)}$: $\{(3,-4,1)\}$.

$$V_{(3)}: \begin{bmatrix} -3 & 0 & 6 \\ 1 & -3 & -11 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

We get a basis of $V_{(3)}$: $\{(2,-3,1)\}$. Hence,

$$\begin{bmatrix} 0 & 0 & 6 \\ 1 & 0 & -11 \\ 0 & 1 & 6 \end{bmatrix}^{2019} = \begin{bmatrix} 6 & 3 & 2 \\ -5 & -4 & -3 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{2019} \cdot \begin{bmatrix} 6 & 3 & 2 \\ -5 & -4 & -3 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

We also get that

$$\begin{bmatrix} 6 & 3 & 2 \\ -5 & -4 & -3 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & -2 & -4 \\ \frac{1}{2} & \frac{3}{2} & \frac{9}{2} \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 0 & 0 & 6 \\ 1 & 0 & -11 \\ 0 & 1 & 6 \end{bmatrix}^{2019} = \begin{bmatrix} 6 & 3 & 2 \\ -5 & -4 & -3 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{2015} & 0 \\ 0 & 0 & 3^{2019} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & -2 & -4 \\ \frac{1}{2} & \frac{3}{2} & \frac{9}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} 6 & 3 \cdot 2^{2015} & 2 \cdot 3^{2019} \\ -5 & -2^{2021} & -3^{2020} \\ 1 & 2^{2019} & 3^{2019} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & -2 & -4 \\ \frac{1}{2} & \frac{3}{2} & \frac{9}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} 3^{2019} - 3 \cdot 2^{2019} + 3 & 3^{2020} - 3 \cdot 2^{2018} + 3 & 3^{2021} - 3 \cdot 2^{2021} + 3 \\ 2^{2021} - \frac{3^{2020}}{2} - \frac{5}{2} & 2^{2022} - \frac{3^{2021}}{2} - \frac{5}{2} & 2^{2023} - \frac{3^{2022}}{2} - \frac{5}{2} \\ \frac{3^{2019}}{2} - 2^{2019} + \frac{1}{2} & \frac{3^{2020}}{2} - 2^{2020} + \frac{1}{2} & \frac{3^{2021}}{2} - 2^{2021} + \frac{1}{2} \end{bmatrix}.$$

Group 9:45

1. Check whether the following matrix A is diagonalizable. If so, find a matrix C such that $C^{-1}AC$ is diagonal.

$$\begin{bmatrix} -3 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & -2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 3 & 4 & 2 \end{bmatrix}$$

   (a) We calculate the characteristic polynomial

$$w(\lambda) = \begin{vmatrix} -3-\lambda & 1 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & -2-\lambda \end{vmatrix} = -\lambda^3 - 4\lambda^2 + 3\lambda + 18 = (\lambda+3)^2(2-\lambda),$$

   So the eigenvalues are -3 and 2. We study the eigenspaces

$$V_{(-3)} : \begin{bmatrix} 0 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   So we get a basis of $V_{(-3)}$: $\{(1, 0, 0)\}$.

$$V_{(2)} : \begin{bmatrix} -5 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{3}{5} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

   Se we get a basis of $V_{(2)}$: $\{(\frac{3}{5}, -2, 1)\}$. Those bases taken together do not give a basis of the whole $\mathbb{R}^3$, so the matrix is not diagonalizable.

   (b) We calculate the characteristic polynomial

$$w(\lambda) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 3 & 4 & 2-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 11a + 6 = -(\lambda-1)(\lambda-2)(\lambda-3).$$

   We get three eigenvalues: $1, 2, 3$. Now, we study eigenspaces

$$V_{(1)} : \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   So we get a basis of $V_{(1)}$: $\{(1, -1, 0)\}$.

$$V_{(2)} : \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   So we get a basis of $V_{(2)}$: $\{(0, 0, 1)\}$.

$$V_{(3)} : \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 3 & 4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{7} \\ 0 & 1 & -\frac{1}{7} \\ 0 & 0 & 0 \end{bmatrix}.$$

   So we get a basis of $V_{(3)}$: $\{(1, 1, 7)\}$. We get a basis of eigenvectors: $\mathcal{A} = \{(1, -1, 0), (0, 0, 1), (1, 1, 7)\}$. Hence,

$$C = M(\mathrm{id})^{st}_{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 7 \end{bmatrix}$$

$$C^{-1}AC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

2. Calculate $\begin{bmatrix} 0 & 0 & -6 \\ -1 & 0 & 11 \\ 0 & -1 & -6 \end{bmatrix}^{2019}$.

The characteristic polynomial is:

$$w(\lambda) = \begin{vmatrix} -\lambda & 0 & -6 \\ -1 & -\lambda & 11 \\ 0 & -1 & -6-\lambda \end{vmatrix} = -\lambda^3 - 6\lambda^2 - 11\lambda - 6 = -(\lambda+1)(\lambda+2)(\lambda+3).$$

We get the eigenvectors $-1, -2, -3$. Now, we study the eigenspaces.

$$V_{(1)} : \begin{bmatrix} 1 & 0 & -6 \\ -1 & 1 & 11 \\ 0 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

So we get a basis of $V_{(1)}$: $\{(6, -5, 1)\}$.

$$V_{(2)} : \begin{bmatrix} 2 & 0 & -6 \\ -1 & 2 & 11 \\ 0 & -1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

So we get a basis of $V_{(2)}$: $\{(3, -4, 1)\}$.

$$V_{(3)} : \begin{bmatrix} 3 & 0 & -6 \\ -1 & 3 & 11 \\ 0 & -1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

So we get a basis of $V_{(3)}$: $\{(2, -3, 1)\}$. Hence,

$$\begin{bmatrix} 0 & 0 & -6 \\ -1 & 0 & 11 \\ 0 & -1 & -6 \end{bmatrix}^{2019} = \begin{bmatrix} 6 & 3 & 2 \\ -5 & -4 & -3 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}^{2019} \cdot \begin{bmatrix} 6 & 3 & 2 \\ -5 & -4 & -3 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

We also get that:

$$\begin{bmatrix} 6 & 3 & 2 \\ -5 & -4 & -3 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & -2 & -4 \\ \frac{1}{2} & \frac{3}{2} & \frac{9}{2} \end{bmatrix}$$

Hence,

$$\begin{bmatrix} 0 & 0 & -6 \\ -1 & 0 & 11 \\ 0 & -1 & -6 \end{bmatrix}^{2019} = \begin{bmatrix} 6 & 3 & 2 \\ -5 & -4 & -3 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2^{2019} & 0 \\ 0 & 0 & -3^{2019} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & -2 & -4 \\ \frac{1}{2} & \frac{3}{2} & \frac{9}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} 6 & -3 \cdot 2^{2019} & -2 \cdot 3^{2019} \\ -5 & 2^{2021} & 3^{2020} \\ 1 & -2^{2019} & -3^{2019} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & -2 & -4 \\ \frac{1}{2} & \frac{3}{2} & \frac{9}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} -3^{2019} + 3 \cdot 2^{2019} + 3 & -3^{2018} + 3 \cdot 2^{2020} + 3 & -3^{2021} + 3 \cdot 2^{2021} + 3 \\ -2^{2021} + \frac{3^{2020}}{2} - \frac{5}{2} & -2^{2023} + \frac{3^{2021}}{2} + \frac{5}{2} & 2^{2023} + \frac{3^{2022}}{2} - \frac{5}{2} \\ -\frac{3^{2019}}{2} + 2^{2019} + \frac{1}{2} & -\frac{3^{2020}}{2} + 2^{2020} + \frac{1}{2} & -\frac{3^{2021}}{2} + 2^{2021} + \frac{1}{2} \end{bmatrix}.$$