Linear algebra, WNE, 2018/2019
meeting 17. – solutions

27 November 2018

1. For endomorphism $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2, \varphi((x,y)) = (3x+4y, 5x-2y)$ and bases $\mathcal{A}_1 = \{(4,1),(3,1)\}, \mathcal{A}_2 = \{(2,3),(5,8)\}, \mathcal{A}_3 = \{(4,2),(1,1)\}$ find matrices $\mathcal{A}_i = M(\varphi)_{\mathcal{A}_i}^{\mathcal{A}_i}$ and matrices C_{ij} satisfying $A_j = C_{ij}^{-1}A_iC_{ij}$ for $i,j=1,2,3$.

   Solution: $M(\varphi)_{\rm st}^{\rm st} = \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix}$. We find the coordinates of vectors from the standard basis in $\mathcal{A}_1$: $(1,0) = (1,-1)_{\mathcal{A}_1}, (0,1) = (-3,4)_{\mathcal{A}_1}$. So $M(\mathrm{id})_{\rm st}^{\mathcal{A}_1} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$ and obviously: $M(\mathrm{id})_{\mathcal{A}_1}^{\rm st} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$. Thus,

$$A_1 = M(\varphi)_{\mathcal{A}_1}^{\mathcal{A}_1} = M(\mathrm{id})_{\rm st}^{\mathcal{A}_1} M(\varphi)_{\rm st}^{\rm st} M(\mathrm{id})_{\mathcal{A}_1}^{\rm st} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -12 & 10 \\ 17 & -12 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -38 & -26 \\ 56 & 39 \end{bmatrix}.$$

   We find the coordinates of vectors from $\mathcal{A}_2$ and $\mathcal{A}_3$ with respect to $\mathcal{A}_1$: $(2,3) = (-7,10)_{\mathcal{A}_1}, (5,8) = (-19,27)_{\mathcal{A}_1}$ and $(4,2) = (-2,4)_{\mathcal{A}_1}, (1,1) = (-2,3)_{\mathcal{A}_1}$ and the coordinates of vectors from $\mathcal{A}_3$ with respect to basis $\mathcal{A}_2$ are $(4,2) = (22,-8)_{\mathcal{A}_2}(1,1) = (3,-1)_{\mathcal{A}_2}$, so

$$C_{12} = M(\mathrm{id})_{\mathcal{A}_2}^{\mathcal{A}_1} = \begin{bmatrix} -7 & -19 \\ 10 & 27 \end{bmatrix},$$

$$C_{13} = M(\mathrm{id})_{\mathcal{A}_3}^{\mathcal{A}_1} = \begin{bmatrix} -2 & -2 \\ 4 & 3 \end{bmatrix},$$

$$C_{23} = M(\mathrm{id})_{\mathcal{A}_3}^{\mathcal{A}_2} = \begin{bmatrix} 22 & 3 \\ -8 & -1 \end{bmatrix},$$

$$C_{21} = C_{12}^{-1} = \begin{bmatrix} 27 & 19 \\ -10 & -7 \end{bmatrix},$$

$$C_{31} = C_{13}^{-1} = \begin{bmatrix} \frac{3}{2} & 1 \\ -2 & -1 \end{bmatrix},$$

$$C_{32} = C_{23}^{-1} = \begin{bmatrix} \frac{-1}{2} & 4 \\ \frac{-3}{2} & 11 \end{bmatrix}.$$

   so

$$A_2 = C_{21}A_1C_{12} = \begin{bmatrix} 27 & 19 \\ -10 & -7 \end{bmatrix} \cdot \begin{bmatrix} -38 & -26 \\ 56 & 39 \end{bmatrix} \cdot \begin{bmatrix} -7 & -19 \\ 10 & 27 \end{bmatrix} = \begin{bmatrix} 38 & 39 \\ -12 & -13 \end{bmatrix} \cdot \begin{bmatrix} -7 & -19 \\ 10 & 27 \end{bmatrix} =$$

$$= \begin{bmatrix} 124 & 331 \\ -46 & -123 \end{bmatrix},$$

$$A_3 = C_{31}A_1C_{13} = \begin{bmatrix} \frac{3}{2} & 1 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} -38 & -26 \\ 56 & 39 \end{bmatrix} \cdot \begin{bmatrix} -2 & -2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 1 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} -28 & -2 \\ 44 & 5 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 2 \\ 12 & -9 \end{bmatrix}.$$

2. For the following endomorphisms find eigenvalues and bases of eigenspaces related to each eigenvalue.

- $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2, \varphi((x,y)) = (2x-y, -x+2y)$,
- $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^4, \varphi((x,y,z,t)) = (-6x-y+2z, 3x+2y+t, -14x-2y+5z, -t)$.

Solution:

- $$w(\lambda) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3,$$

  so $\Delta = 16 - 12 = 4$, so the eigenvalues are $1, 3$.

  The eigenspace related to eigenvalue 1 is described by the system $\begin{cases} x - y = 0 \\ -x + y = 0 \end{cases}$, so its basis has only one vector: $(1,1)$.

  The eigenspace related to eigenvalue 3 is described by $\begin{cases} -x - y = 0 \\ -x - y = 0 \end{cases}$, so its basis has only one vector: $(1,-1)$.

- $$w(\lambda) = \begin{vmatrix} -6-\lambda & -1 & 2 & 0 \\ 3 & 2-\lambda & 0 & 1 \\ -14 & -2 & 5-\lambda & 0 \\ 0 & 0 & 0 & -1-\lambda \end{vmatrix} =$$

  $$= (-1-\lambda)\begin{vmatrix} -6-\lambda & -1 & 2 \\ 3 & 2-\lambda & 0 \\ -14 & -2 & 5-\lambda \end{vmatrix} = (\lambda+1)(\lambda^3 - \lambda^2 - \lambda + 1) = (\lambda-1)^2(\lambda+1)^2.$$

  So the eigenvalues are 1 and -1.
  The eigenspace for eigenvalue 1:

  $$\begin{bmatrix} -7 & -1 & 2 & 0 \\ 3 & 1 & 0 & 1 \\ -14 & -2 & 4 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{w_2 \cdot 7} \begin{bmatrix} -7 & -1 & 2 & 0 \\ 21 & 7 & 0 & 7 \\ -14 & -2 & 4 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{w_2 + 3w_1, w_3 - 2w_1, w_4 \cdot \frac{-1}{2}}$$

  $$\begin{bmatrix} -7 & -1 & 2 & 0 \\ 0 & 4 & 6 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{w_2 - 7w_4} \begin{bmatrix} -7 & -1 & 2 & 0 \\ 0 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{1}{4}}$$

  $$\begin{bmatrix} -7 & -1 & 2 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{w_1 + w_2} \begin{bmatrix} -7 & 0 & \frac{7}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{w_1 \cdot -17} \begin{bmatrix} 1 & 0 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  so a basis contains one vector: $(1,-3,2,0)$.
  The eigenspace for eigenvalue -1:

  $$\begin{bmatrix} -5 & -1 & 2 & 0 \\ 3 & 3 & 0 & 1 \\ -14 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 \cdot 5, w_3 \cdot 5} \begin{bmatrix} -5 & -1 & 2 & 0 \\ 15 & 15 & 0 & 5 \\ -70 & -10 & 30 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 + 3w_1, w_3 - 14w_1}$$

  $$\begin{bmatrix} -5 & -1 & 2 & 0 \\ 0 & 12 & 6 & 5 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 \leftrightarrow w_3} \begin{bmatrix} -5 & -1 & 2 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 12 & 6 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_3 - 3w_2}$$

$$\begin{bmatrix} -5 & -1 & 2 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{1}{4}, w_3 \cdot \frac{1}{5}} \begin{bmatrix} -5 & -1 & 2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1 + w_2}$$

$$\begin{bmatrix} -5 & 0 & \frac{5}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1 \cdot \frac{-1}{5}} \begin{bmatrix} 1 & 0 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so a basis contains one vector $(1, -1, 2, 0)$.

3. For the endomorphism $\varphi\colon V \to V$ check whether there exists a basis $\mathcal{A}$ of V which consists of eigenvectors of φ. If the answer is in the positive, give an example of such a basis and calculate $M(\varphi)_{\mathcal{A}}^{\mathcal{A}}$.

- $V = \mathbb{R}^2, \varphi((a, b)) = (a - b, a + 3b)$,
- $V = \mathbb{R}^4, \varphi((a, b, c, d)) = (2a + 4b, 5a + 3b, c + d, 3c - d)$.

Solution:

- $$w(\lambda) = (1 - \lambda)(3 - \lambda) + 1 = \lambda^2 - 4\lambda + 4,$$

  so we get only one eigenvalue, 2. The eigenspace for 2 is spanned by $(1, -1)$, so the space spanned by all eigenvectors has only one dimension, which is not enough.

- $$w(\lambda) = ((2 - \lambda)(3 - \lambda) - 20)((1 - \lambda)(-1 - \lambda) - 3) = (\lambda^2 - 5\lambda - 14)(\lambda^2 - 4)$$

  So we get three eigenvalues $-2, 2$ and 7.
  For -2:

$$\begin{bmatrix} 4 & 4 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix} \xrightarrow{w_2 - \frac{5}{4}w_1, w_4 - w_3} \begin{bmatrix} 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1 \cdot 14, w_2 \leftrightarrow w_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  so a basis of eigenspace is $(1, -1, 0, 0), (0, 0, 1, -3)$.
  For 2:

$$\begin{bmatrix} 0 & 4 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 3 & -3 \end{bmatrix} \xrightarrow{w_1 \cdot \frac{1}{4}, w_4 - 3w_3} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 - w_1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{1}{5}}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_2, w_3 \cdot (-1)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  so a basis of eigenspace is $(0, 0, 1, 1)$.
  For 7:

$$\begin{bmatrix} -5 & 4 & 0 & 0 \\ 5 & -4 & 0 & 0 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 3 & -8 \end{bmatrix}$$

  so a basis of eigenspace is $(4, 5, 0, 0)$.
  Together those vectors constitute a basis of V: $\mathcal{A} = \{(1, -1, 0, 0), (0, 0, 1, -3), (0, 0, 1, 1), (4, 5, 0, 0)\}$ and $M(\varphi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

4. Check whether matrices $A_1 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$ are diagonalizable. If so, find C_i, such that $C_i^{-1}A_iC_i$ is diagonal for $i = 1, 2$.

   Solution: $w_1(\lambda) = (1-\lambda)(3-\lambda)+1 = \lambda^2 - 4\lambda + 4$. We get only one eigenvalue 2, and the eigenspace is spanned by $(1, 1)$, so we there is no basis consisting eigenvectors, and A_1 is not diagonalizable.
   $w_2(\lambda) = (5-\lambda)(-1-\lambda)+9 = \lambda^2 - 4\lambda + 4$. Similarly, we have only one eigenvalue 2, and the eigenspace is spanned by only one vector: $(1, 1)$, so again there is no basis consisting of eigenvectors and the matrix is not diagonalizable.