Linear algebra, WNE, 2018/2019
meeting 17. – homework solutions

27 November 2018

Group 8:00

1. For the eigenvalues and bases of their eigenspaces of endomorphism $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3, \varphi((x,y,z)) = (x-y, x+3y+z, 2z)$.

   We calculate the characteristic polynomial:

$$w(\lambda) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ 1 & 3-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = (2-\lambda)^3.$$

   So the only eigenvalue is 2. Let us find a basis of $V_{(2)}$:

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   A basis is $\{(1,-1,0)\}$.

2. Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3, \varphi((a,b,c)) = (3a, a+2b-c, a-b+2c)$. Check whether there exists a basis $\mathcal{A}$ of $\mathbb{R}^3$ which consists of eigenvectors of φ. If the answer is in the positive, give an example of such a basis and find $M(\varphi)_{\mathcal{A}}^{\mathcal{A}}$.

   First we calculate the characteristic polynomial

$$w(\lambda) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = -a^3 + 7a^2 - 15a + 9 = (1-a)(a-3)^2.$$

   So the eigenvalues are 1 and 3. Let us find a basis of $V_{(1)}$:

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

   So $\{(0,1,1)\}$ is a basis. Let us analyse $V_{(3)}$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   So $\{(1,1,0),(1,0,1)\}$ is a basis. Together bases of $V_{(1)}$ and $V_{(3)}$ give a basis of the whole space so $\mathcal{A} = \{(0,1,1),(1,1,0),(1,0,1)\}$ is a basis consisting of eigenvectors, and so:

$$M(\varphi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

3. Check whether matrix $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ is diagonalizable. If it is, find a matrix C, such that $C^{-1}AC$ is a diagonal matrix.

   We start with the characteristic polynomial

$$w(\lambda) = \begin{vmatrix} 4-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6,$$

   so 2 and 3 are the eigenvalues.
   $V_{(2)}$ is one-dimensional, and a basis is $(1,-1)$, and a basis of $V_{(3)}$ is $(-2,1)$, so basis $\mathcal{A} = \{(1,-1),(-2,1)\}$ consists of eigenvectors and $M(id)_{st}^{\mathcal{A}} A M(\mathrm{id})_{\mathcal{A}}^{st} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is diagonal. Hence, $C = M(\mathrm{id})_{\mathcal{A}}^{st} = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$.

Group 9:45

1. For the eigenvalues and bases of their eigenspaces of endomorphism $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3, \varphi((x,y,z)) = (x - z, 2y, x+y+3z)$.

   We start with the characteristic polynomial

$$w(\lambda) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 3-\lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 + 5\lambda - 8 = (2-\lambda)^3.$$

   So the only eigenvalue is 2. Let's find a basis of $V_{(2)}$:

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   So the basis is $\{(1,0,-1)\}$.

2. Consider endomorphism $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3, \varphi((a,b,c)) = (2a-b+c, -a+2b+c, 3c)$. Check whether there exists a a basis $\mathcal{A}$ of $\mathbb{R}^3$ which consists of eigenvectors of φ. If the answer is in the positive, give an example of such a basis and find $M(\varphi)_{\mathcal{A}}^{\mathcal{A}}$.

   We start with the characteristic polynomial

$$w(\lambda) = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = -a^3 + 7a^2 - 15a + 9 = (1-a)(a-3)^2.$$

   So 1 and 3 are the eigenvalues. We calculate a basis of $V_{(1)}$:

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   So $\{(1,1,0)\}$ is a basis. Let us check $V_{(3)}$:

$$\begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   We get a basis $\{(1,-1,0),(1,0,1)\}$. Together bases of $V_{(1)}$ and $V_{(3)}$ give a basis of the whole space $\mathcal{A} = \{(1,1,0),(1,-1,0),(1,0,1)\}$ and:

$$M(\varphi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

3. Check whether $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ is diagonalizable. If it is, find a matrix C, such that $C^{-1}AC$ is a diagonal matrix.

We start by calculating the characteristic polynomial

$$w(\lambda) = \begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6,$$

So the eigenvalues are 2 and 3.
$V_{(2)}$ has only one vector in its basis $(1,2)$, and for $V_{(3)}$ we get $(1,1)$, so $\mathcal{A} = \{(1,2),(1,1)\}$ is a basis consisting of eigenvectors and $M(id)_{st}^{\mathcal{A}} A M(\mathrm{id})_{\mathcal{A}}^{st} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is diagonal. So $C = M(\mathrm{id})_{\mathcal{A}}^{st} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.