Linear algebra, WNE, 2018/2019
meeting 14. – solutions

20 November 2018

1. Find the determinants of the following matrices
$\begin{bmatrix} 1 & 2 \\ -1 & -5 \end{bmatrix}$, $\begin{bmatrix} 2 & -1 & 3 \\ 4 & 5 & -1 \\ 2 & 6 & -4 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & 0 & 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & 1 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 3 & 4 & 5 & 4 \end{bmatrix}$.

   We get
$$\begin{vmatrix} 1 & 2 \\ -1 & -5 \end{vmatrix} = -5 - (-2) = -3,$$
$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 5 & -1 \\ 2 & 6 & -4 \end{vmatrix} = -40 + 2 + 72 - 30 - 16 + 12 = 0,$$
$$\begin{vmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & 0 & 2 & 0 \end{vmatrix} = 0 \cdot (-1)^{2+1} + 1 \cdot (-1)^{2+2} \begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{vmatrix} + 1 \cdot (-1)^{2+3} \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} + 0 \cdot (-1)^{2+4} =$$
$$= (0 + 1 + 12 - 9 - 2 - 0) - (0 + 2 + 0 - 3 - 0 - 0) = 2 - (-1) = 3,$$
$$\begin{vmatrix} 2 & 1 & 1 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 3 & 4 & 5 & 4 \end{vmatrix} = 0 \cdot (-1)^{3+1} + 3 \cdot (-1)^{3+2} \begin{vmatrix} 2 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 5 & 4 \end{vmatrix} + 1 \cdot (-1)^{3+3} \begin{vmatrix} 2 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 4 \end{vmatrix} + 0 \cdot (-1)^{3+4} =$$
$$= -3 \cdot (-13) + (-9) = 30.$$

2. Let $A = \begin{bmatrix} 1 & r & 1 & 0 \\ 0 & 1 & r & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. For which real numbers $r \in \mathbb{R}$, $\det A = 1$?

   $\det A = 1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & r & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} + 0 + 1 \cdot (-1)^{3+1} \begin{vmatrix} r & 1 & 0 \\ 1 & r & 1 \\ 0 & 1 & 1 \end{vmatrix} + 0 = (1 - r) + (r^2 - r - 1) = r^2 - 2r$. And $r^2 - 2r = 1$ for $r = 1 \pm \sqrt{2}$.

3. Calculate the determinant of $\begin{bmatrix} 3 & 4 & 2 & 2 \\ 4 & 5 & 6 & 5 \\ 2 & 3 & 6 & 0 \\ 8 & 7 & 10 & 18 \end{bmatrix}$, by transforming it to echelon form.

$$\begin{bmatrix} 3 & 4 & 2 & 2 \\ 4 & 5 & 6 & 5 \\ 2 & 3 & 6 & 0 \\ 8 & 7 & 10 & 18 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_3} \begin{bmatrix} 2 & 3 & 6 & 0 \\ 4 & 5 & 6 & 5 \\ 3 & 4 & 2 & 2 \\ 8 & 7 & 10 & 18 \end{bmatrix} \xrightarrow{w_3 \cdot 2} \begin{bmatrix} 2 & 3 & 6 & 0 \\ 4 & 5 & 6 & 5 \\ 6 & 8 & 4 & 4 \\ 8 & 7 & 10 & 18 \end{bmatrix} \xrightarrow{w_2 - 2w_1, w_3 - 3w_1, w_4 - 4w_1}$$

$$\begin{bmatrix} 2 & 3 & 6 & 0 \\ 0 & -1 & -6 & 5 \\ 0 & -1 & -14 & 4 \\ 0 & -5 & -14 & 18 \end{bmatrix} \xrightarrow{w_3 - w_2, w_4 - 5w_2} \begin{bmatrix} 2 & 3 & 6 & 0 \\ 0 & -1 & -6 & 5 \\ 0 & 0 & -8 & -1 \\ 0 & 0 & 16 & -7 \end{bmatrix} \xrightarrow{w_4 + 2w_3} \begin{bmatrix} 2 & 3 & 6 & 0 \\ 0 & -1 & -6 & 5 \\ 0 & 0 & -8 & -1 \\ 0 & 0 & 0 & -9 \end{bmatrix}$$

So the determinant of the final matrix is $2 \cdot (-1) \cdot (-8) \cdot (-9) = -144$. We have swapped the rows once, and one of the rows was multiplied by 2, so the result has to be multiplied by (-1) and $\frac{1}{2}$. Hence, we get $-144 \cdot (-1) \cdot \frac{1}{2} = 72$.

4. Calculate the determinant of $\begin{bmatrix} 3 & 2 & 1 & 6 & 5 & -1 \\ -1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 4 & 9 & 11 & -1 \\ 0 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 1 & 3 \end{bmatrix}$.

$$\begin{vmatrix} 3 & 2 & 1 & 6 & 5 & -1 \\ -1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 4 & 9 & 11 & -1 \\ 0 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & 4 \end{vmatrix} \cdot \begin{vmatrix} 3 & -1 & 0 \\ -1 & 0 & 2 \\ 2 & 1 & 3 \end{vmatrix} = 1 \cdot (-13) = -13$$