Linear algebra, WNE, 2018/2019
meeting 13., solutions

15 November 2018

1. Let $\mathcal{A} = \{(-2,1),(-1,1)\}, \mathcal{B} = \{(3,2),(2,-2)\}$, $\mathcal{C} = \{(1,0,1,0),(0,0,-1,0),(0,2,0,1),(0,1,0,1)\}$, and let $\phi, \varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ i $\psi\colon \mathbb{R}^2 \to \mathbb{R}^4$ be such that

   - $\psi((x,y)) = (x+y, -x, -3y, -x+2y)$,
   - $M(\phi)^{st}_{\mathcal{A}} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$,
   - $M(\varphi)^{\mathcal{B}}_{\mathcal{A}} = \begin{bmatrix} -1 & 0 \\ -2 & 3 \end{bmatrix}$,

   Find

   - $M(\mathrm{id})^{\mathcal{C}}_{st}$,
     We see that
     - $(1,0,0,0) = 1(1,0,1,0) + 1(0,0,-1,0) + 0(0,2,0,1) + 0(0,1,0,1)$,
     - $(0,1,0,0) = 0(1,0,1,0) + 0(0,0,-1,0) + 1(0,2,0,1) - 1(0,1,0,1)$,
     - $(0,0,1,0) = 0(1,0,1,0) - 1(0,0,-1,0) + 0(0,2,0,1) + 0(0,1,0,1)$,
     - $(0,0,0,1) = 0(1,0,1,0) + 0(0,0,-1,0) - 1(0,2,0,1) + 2(0,1,0,1)$.

     Therefore,
     $$M(\mathrm{id})^{\mathcal{C}}_{st} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 2 \end{bmatrix}.$$
   - $M(\psi)^{st}_{st}$,
     We write out the matrix from the formula $M(\psi)^{st}_{st} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -3 \\ -1 & 2 \end{bmatrix}$.
   - $M(\varphi)^{st}_{\mathcal{A}}$,
     We have $M(\varphi)^{st}_{\mathcal{A}} = M(\mathrm{id})^{st}_{\mathcal{B}} \cdot M(\varphi)^{\mathcal{B}}_{\mathcal{A}} = \begin{bmatrix} 3 & 2 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -7 & 6 \\ 2 & -6 \end{bmatrix}$.
   - $M(\psi \circ (\varphi + 3\phi))^{\mathcal{C}}_{\mathcal{A}}$,
     We get:
     $$M(\psi \circ (\varphi + 3\phi))^{\mathcal{C}}_{\mathcal{A}} = M(\psi)^{\mathcal{C}}_{st} \cdot M(\varphi + 3\phi)^{st}_{\mathcal{A}} = M(\mathrm{id})^{\mathcal{C}}_{st} \cdot M(\psi)^{st}_{st} \cdot \left(M(\varphi)^{st}_{\mathcal{A}} + 3M(\phi)^{st}_{\mathcal{A}}\right) =$$
     $$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -3 \\ -1 & 2 \end{bmatrix} \cdot \left(\begin{bmatrix} -7 & 6 \\ 2 & -6 \end{bmatrix} + 3 \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \right) =$$
     $$= \begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 0 & -2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -4 & 9 \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 28 & -15 \\ -16 & 12 \\ 36 & -33 \end{bmatrix}$$

- the coordinates of $\psi(\varphi(v)+3\phi(v))$ in the basis $\mathcal{C}$, if vector v has coordinates $1,1$ with respect to $\mathcal{A}$. The coordinates are

$$M(\psi\circ(\varphi+3\phi))_{\mathcal{A}}^{\mathcal{C}}\cdot\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}7\\13\\-4\\3\end{bmatrix},$$

so $7,13,-4,3$.

2. Let $\mathcal{A}=\{(5,7,1),(4,0,0),(6,2,5)\},\mathcal{B}=\{(1,-1,1),(0,1,6),(0,1,5)\}$. Find a matrix $C\in M_{3\times 3}(\mathbb{R})$ such that for every $\alpha\in\mathbb{R}^3$ we get the following. If a_1,a_2,a_3 are the coordinates of α with respect to $\mathcal{A}$, and b_1,b_2,b_3 are the coordinates of this vector with respect to $\mathcal{B}$, then

$$C\cdot\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

Notice that C equals $M(\mathrm{id})_{\mathcal{A}}^{\mathcal{B}}$. So in the subsequent columns we have to write coordinates of subsequent vectors from $\mathcal{A}$ with respect to $\mathcal{B}$. we find them first:

$$\begin{bmatrix}1&0&0&5&4&6\\-1&1&1&7&0&2\\1&6&5&1&0&5\end{bmatrix}\xrightarrow{w_2+w_1,w_3-w_1}\begin{bmatrix}1&0&0&5&4&6\\0&1&1&12&4&8\\0&6&5&-4&-4&-1\end{bmatrix}\xrightarrow{w_3-6w_2}$$

$$\begin{bmatrix}1&0&0&5&4&6\\0&1&1&12&4&8\\0&0&-1&-76&-28&-49\end{bmatrix}\xrightarrow{w_2+w_3}\begin{bmatrix}1&0&0&5&4&6\\0&1&0&-64&-24&-41\\0&0&-1&-76&-28&-49\end{bmatrix}\xrightarrow{w_3\cdot(-1)}$$

$$\begin{bmatrix}1&0&0&5&4&6\\0&1&0&-64&-24&-41\\0&0&1&76&28&49\end{bmatrix}$$

So,

$$C=M(\mathrm{id})_{\mathcal{A}}^{\mathcal{B}}=\begin{bmatrix}5&4&6\\-64&-24&-41\\76&28&49\end{bmatrix}$$

3. Let $\mathcal{A}=\{(2,1),(1,1)\},\mathcal{B}=\{(1,3),(0,1)\},\mathcal{C}=\{(0,1),(1,4)\}$, and let $\varphi\colon\mathbb{R}^2\to\mathbb{R}^2$ be a linear transformation such that $M(\varphi)_{\mathcal{A}}^{\mathcal{B}}=\begin{bmatrix}1&2\\3&4\end{bmatrix}$. Find $M(\varphi)_{\mathcal{A}}^{\mathcal{C}}$.

$M(\varphi)_{\mathcal{A}}^{\mathcal{C}}=M(\mathrm{id})_{\mathcal{B}}^{\mathcal{C}}\cdot M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$, so we need $M(\mathrm{id})_{\mathcal{B}}^{\mathcal{C}}$, so we need to calculate the coordinates of vectors $\mathcal{B}$ with respect to $\mathcal{C}$. We get $(1,3)=-(0,1)+(1,4)$ and $(0,1)=(0,1)+0(1,4)$, so

$$M(\mathrm{id})_{\mathcal{B}}^{\mathcal{C}}=\begin{bmatrix}-1&1\\1&0\end{bmatrix},$$

hence,

$$M(\varphi)_{\mathcal{A}}^{\mathcal{C}}=M(\mathrm{id})_{\mathcal{B}}^{\mathcal{C}}\cdot M(\varphi)_{\mathcal{A}}^{\mathcal{B}}=\begin{bmatrix}-1&1\\1&0\end{bmatrix}\cdot\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}2&2\\1&2\end{bmatrix}.$$