Linear algebra, WNE, 2018/2019
meeting 13. – homework solutions

15 November 2018

Group 8:00

1. Let $\mathcal{A} = \{(1,2,3),(2,1,0),(4,5,0)\}, \mathcal{B} = \{(2,1,2),(3,1,2),(2,1,3)\}$. Find a matrix $C \in M_{3\times 3}(\mathbb{R})$, such that for any vector $\alpha \in \mathbb{R}^3$ we get that if the coordinates of α with respect to $\mathcal{A}$ are a_1, a_2, a_3 , and b_1, b_2, b_3 are the coordinates of this vector with respect to $\mathcal{B}$, then
$$C \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

   Matrix C simply equals to $M(\mathrm{id})^{\mathcal{B}}_{\mathcal{A}}$, so we have to find the coordinates of vectors from $\mathcal{A}$ with respect to $\mathcal{B}$:
$$\begin{bmatrix} 2&3&2&1&2&4\\ 1&1&1&2&1&5\\ 2&2&3&3&0&0 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_2} \begin{bmatrix} 1&1&1&2&1&5\\ 2&3&2&1&2&4\\ 2&2&3&3&0&0 \end{bmatrix} \xrightarrow{w_2-2w_1, w_3-2w_1}$$
$$\begin{bmatrix} 1&1&1&2&1&5\\ 0&1&0&-3&0&-6\\ 0&0&1&-1&-2&-10 \end{bmatrix} \xrightarrow{w_1-w_3} \begin{bmatrix} 1&1&0&3&3&15\\ 0&1&0&-3&0&-6\\ 0&0&1&-1&-2&-10 \end{bmatrix} \xrightarrow{w_1-w_2}$$
$$\begin{bmatrix} 1&0&0&6&3&21\\ 0&1&0&-3&0&-6\\ 0&0&1&-1&-2&-10 \end{bmatrix}$$

   Therefore, $M(\mathrm{id})^{\mathcal{B}}_{\mathcal{A}} = C = \begin{bmatrix} 6&3&21\\ -3&0&-6\\ -1&-2&-10 \end{bmatrix}$.

2. Let $\mathcal{A} = \{(1,1,2),(1,2,1),(0,0,1)\}, \mathcal{B} = \{(2,1),(1,-1)\}, \mathcal{C} = \{(2,4),(-1,1)\}$ and let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $M(\varphi)^{\mathcal{B}}_{\mathcal{A}} = \begin{bmatrix} -1&3&2\\ 5&-3&4 \end{bmatrix}$. Find $M(\varphi)^{\mathcal{C}}_{\mathcal{A}}$.

   We know that $M(\varphi)^{\mathcal{C}}_{\mathcal{A}} = M(\mathrm{id})^{\mathcal{C}}_{\mathcal{B}} \cdot M(\varphi)^{\mathcal{B}}_{\mathcal{A}}$.

   Notice that $(1,-1) = -(-1,1)$ and $(2,1) = \frac{1}{2}(2,4) - (-1,1)$, hence, $M(\mathrm{id})^{\mathcal{C}}_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & 0\\ -1 & -1 \end{bmatrix}$. Therefore,
$$M(\varphi)^{\mathcal{C}}_{\mathcal{A}} = \begin{bmatrix} \frac{1}{2} & 0\\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1&3&2\\ 5&-3&4 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{3}{2} & 1\\ -4 & 0 & -6 \end{bmatrix}.$$

Group 9:00

1. Let $\mathcal{A} = \{(2,1,0),(1,2,3),(4,5,0)\}, \mathcal{B} = \{(1,2,2),(1,2,3),(1,3,2)\}$. Find a matrix $C \in M_{3\times 3}(\mathbb{R})$, such that for any vector $\alpha \in \mathbb{R}^3$ we get the following. If a_1, a_2, a_3 are the coordinates of α with respect to $\mathcal{A}$, and b_1, b_2, b_3 are the coordinates of this vector with respect to $\mathcal{B}$, then
$$C \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Matrix C simply equals to $M(\mathrm{id})_{\mathcal{A}}^{\mathcal{B}}$, so we have to find the coordinates of vectors from $\mathcal{A}$ with respect to $\mathcal{B}$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 1 & 4 \\ 2 & 2 & 3 & 1 & 2 & 5 \\ 2 & 3 & 2 & 0 & 3 & 0 \end{array}\right] \xrightarrow{w_2 - 2w_1,\, w_3 - 2w_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & -3 & 0 & -3 \\ 0 & 1 & 0 & -4 & 1 & -8 \end{array}\right] \xrightarrow{w_3 \leftrightarrow w_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -4 & 1 & -8 \\ 0 & 0 & 1 & -3 & 0 & -3 \end{array}\right] \xrightarrow{w_1 - w_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 5 & 1 & 7 \\ 0 & 1 & 0 & -4 & 1 & -8 \\ 0 & 0 & 1 & -3 & 0 & -3 \end{array}\right] \xrightarrow{w_1 - w_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & 0 & 15 \\ 0 & 1 & 0 & -4 & 1 & -8 \\ 0 & 0 & 1 & -3 & 0 & -3 \end{array}\right]$$

Thus, $M(\mathrm{id})_{\mathcal{A}}^{\mathcal{B}} = C = \left[\begin{array}{ccc} 9 & 0 & 15 \\ -4 & 1 & -8 \\ -3 & 0 & -3 \end{array}\right]$.

2. Let $\mathcal{A} = \{(2,1,1),(1,2,1),(1,0,0)\}, \mathcal{B} = \{(1,-1),(2,1)\}, \mathcal{C} = \{(-1,1),(2,4)\}$ and let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \left[\begin{array}{ccc} 1 & 3 & 2 \\ 5 & 3 & 4 \end{array}\right]$. Find $M(\varphi)_{\mathcal{A}}^{\mathcal{C}}$.

   We know that $M(\varphi)_{\mathcal{A}}^{\mathcal{C}} = M(\mathrm{id})_{\mathcal{B}}^{\mathcal{C}} \cdot M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$.
   Notice that $(1,-1) = -(-1,1)$ and $(2,1) = -(-1,1) + \frac{1}{2}(2,4)$, therefore $M(\mathrm{id})_{\mathcal{B}}^{\mathcal{C}} = \left[\begin{array}{cc} -1 & -1 \\ 0 & \frac{1}{2} \end{array}\right]$. Hence,

$$M(\varphi)_{\mathcal{A}}^{\mathcal{C}} = \left[\begin{array}{cc} -1 & -1 \\ 0 & \frac{1}{2} \end{array}\right] \cdot \left[\begin{array}{ccc} 1 & 3 & 2 \\ 5 & 3 & 4 \end{array}\right] = \left[\begin{array}{ccc} -6 & -6 & -6 \\ \frac{5}{2} & \frac{3}{2} & 2 \end{array}\right].$$