Linear algebra, WNE, 2018/2019
meeting 12. – solutions

13 November 2018

1. Find the matrices of a linear transformation φ in the standard bases and also in bases $\mathcal{A}, \mathcal{B}$:

   - $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2, \varphi((x_1, x_2, x_3)) = (x_1 - x_2 + 4x_3, -3x_1 + 8x_3), \mathcal{A} = \{(3,4,1),(2,3,1),(5,1,1)\}, \mathcal{B} = \{(3,1),(2,1)\}$,
   - $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^4, \varphi((x,y)) = (3x+y, x+5y, -x+4y, 2x+y), \mathcal{A} = \{(3,1),(4,2)\}$, $\mathcal{B} = \{(1,0,1,0),(0,1,1,1),(0,1,2,3),(0,0,0,1)\}$,
   - $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3, \varphi((x,y,z)) = (4x+y+z, 3x+2y+z, 3x+2y+z), \mathcal{A} = \{(3,1,1),(1,0,0),(5,1,0)\}, \mathcal{B} = \{(1,-1,1),(4,1,1),(2,0,1)\}$.

   - In the standard bases:
   $$M(\varphi)^{\text{st}}_{\text{st}} = \begin{bmatrix} 1 & -1 & 4 \\ -3 & 0 & 8 \end{bmatrix}$$
   We calculate the values for vectors from $\mathcal{A}$: $\varphi((3,4,1)) = (3-4+4, -9+8) = (3,-1)$, $\varphi((2,3,1)) = (2-3+4, -6+8) = (3,2)$ and $\varphi((5,1,1)) = (5-1+4, -15+8) = (8,-7)$. And their coordinates with rescepct to $\mathcal{B}$: $(3,-1) = 5(3,1) - 6(2,1)$, $(3,2) = -(3,2) + 3(2,1)$ and $(8,-7) = 22(3,1) - 29(2,1)$, so
   $$M(\varphi)^{\mathcal{B}}_{\mathcal{A}} = \begin{bmatrix} 5 & -1 & 22 \\ -6 & 3 & -29 \end{bmatrix}.$$

   - In the standard bases:
   $$M(\varphi)^{\text{st}}_{\text{st}} = \begin{bmatrix} 3 & 1 \\ 1 & 5 \\ -1 & 4 \\ 2 & 1 \end{bmatrix}$$
   We calculate the values of vectors from $\mathcal{A}$: $\varphi((3,1)) = (9+1, 3+5, -3+4, 6+1) = (10,8,1,7)$ i $\varphi((4,2)) = (12+2, 4+10, -4+8, 8+2) = (14,14,4,10)$. And their coordinates with respect to $\mathcal{B}$:
   $$\begin{bmatrix} 1 & 0 & 0 & 0 & 10 & 14 \\ 0 & 1 & 1 & 0 & 8 & 14 \\ 1 & 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 3 & 1 & 7 & 10 \end{bmatrix} \xrightarrow{w_3 - w_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 10 & 14 \\ 0 & 1 & 1 & 0 & 8 & 14 \\ 0 & 1 & 2 & 0 & -9 & -10 \\ 0 & 1 & 3 & 1 & 7 & 10 \end{bmatrix} \xrightarrow{w_3 - w_2, w_4 - w_2}$$
   $$\begin{bmatrix} 1 & 0 & 0 & 0 & 10 & 14 \\ 0 & 1 & 1 & 0 & 8 & 14 \\ 0 & 0 & 1 & 0 & -17 & -24 \\ 0 & 0 & 2 & 1 & -1 & -4 \end{bmatrix} \xrightarrow{w_4 - 2w_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 10 & 14 \\ 0 & 1 & 1 & 0 & 8 & 14 \\ 0 & 0 & 1 & 0 & -17 & -24 \\ 0 & 0 & 0 & 1 & 33 & 44 \end{bmatrix} \xrightarrow{w_2 - w_3}$$
   $$\begin{bmatrix} 1 & 0 & 0 & 0 & 10 & 14 \\ 0 & 1 & 0 & 0 & 25 & 38 \\ 0 & 0 & 1 & 0 & -17 & -24 \\ 0 & 0 & 0 & 1 & 33 & 44 \end{bmatrix}$$
   Thus,
   $$M(\varphi)^{\mathcal{B}}_{\mathcal{A}} = \begin{bmatrix} 10 & 14 \\ 25 & 38 \\ -17 & -24 \\ 33 & 44 \end{bmatrix}.$$

- In the standard bases:

$$M(\varphi)_{\mathrm{st}}^{\mathrm{st}} = \begin{bmatrix} 4 & 1 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

We calculate the values for vectors from $\mathcal{A}$: $\varphi((3,1,1)) = (12+1+1, 9+2+1, 9+2+1) = (14,12,12)$, $\varphi((1,0,0)) = (4,3,3)$ and $\varphi((5,1,0)) = (20+1, 15+2, 15+2) = (21,17,17)$. And their coordinates with respect to $\mathcal{B}$:

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 14 & 4 & 21 \\ -1 & 1 & 0 & 12 & 3 & 17 \\ 1 & 1 & 1 & 12 & 3 & 17 \end{array}\right] \xrightarrow{w_2+w_1,\, w_3-w_1} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 14 & 4 & 21 \\ 0 & 5 & 2 & 26 & 7 & 38 \\ 0 & -3 & -1 & -2 & -1 & -4 \end{array}\right] \xrightarrow{w_3 \cdot 5}$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 14 & 4 & 21 \\ 0 & 5 & 2 & 26 & 7 & 38 \\ 0 & -15 & -5 & -10 & -5 & -20 \end{array}\right] \xrightarrow{w_3 - 3w_2} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 14 & 4 & 21 \\ 0 & 5 & 2 & 26 & 7 & 38 \\ 0 & 0 & 1 & 68 & 16 & 94 \end{array}\right] \xrightarrow{w_1 - 2w_3,\, w_2 - 2w_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 0 & -122 & -28 & -167 \\ 0 & 5 & 0 & -110 & -25 & -150 \\ 0 & 0 & 1 & 68 & 16 & 94 \end{array}\right] \xrightarrow{w_2 \cdot \frac{1}{5}} \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & -122 & -28 & -167 \\ 0 & 1 & 0 & -22 & -5 & -30 \\ 0 & 0 & 1 & 68 & 16 & 94 \end{array}\right] \xrightarrow{w_1 - 4w_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -34 & -8 & -47 \\ 0 & 1 & 0 & -22 & -5 & -30 \\ 0 & 0 & 1 & 68 & 16 & 94 \end{array}\right]$$

Thus,

$$M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} -34 & -8 & -47 \\ -22 & -5 & -30 \\ 68 & 16 & 94 \end{bmatrix}.$$

2. Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with matrix

$$M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

in bases $\mathcal{A} = \{(3,1,1), (1,0,0), (5,1,0)\}$, $\mathcal{B} = \{(3,4,5), (4,1,1), (2,0,1)\}$. Find the formula for φ.

The coordinated of the vectors from the standard basis with respect to $\mathcal{A}$ are:
$(1,0,0) = (0,1,0)_{\mathcal{A}}$, $(0,1,0) = (0,-5,1)_{\mathcal{A}}$ and $(0,0,1) = (1,2,-1)_{\mathcal{A}}$. Thus,

$$M(\mathrm{id})_{\mathrm{st}}^{\mathcal{A}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -5 & 2 \\ 0 & 1 & -1 \end{bmatrix},$$

But, obviously

$$M(\mathrm{id})_{\mathcal{B}}^{\mathrm{st}} = \begin{bmatrix} 3 & 4 & 2 \\ 4 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}.$$

Thus,

$$M(\varphi)_{\mathrm{st}}^{\mathrm{st}} = M(\mathrm{id})_{\mathcal{B}}^{st} \cdot M(\varphi)_{\mathcal{A}}^{\mathcal{B}} \cdot M(\mathrm{id})_{\mathrm{st}}^{\mathcal{A}} = \begin{bmatrix} 3 & 4 & 2 \\ 4 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 4 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 1 & -5 & 2 \\ 0 & 1 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 & 4 & 2 \\ 4 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -19 & 3 \\ 5 & -6 & -3 \\ 7 & -11 & -3 \end{bmatrix}.$$

Therefore, $\varphi((x,y,z)) = (9x - 19y + 3z, 5x - 6y - 3z, 7x - 11y - 3z)$.

3. Let $\varphi\colon V \to W, \psi\colon W \to Z$ be linear transformations with $M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 2 & 1 & 4 & 5 \\ 1 & 0 & 1 & 3 \end{bmatrix}$ and $M(\psi)_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 0 & 1 \end{bmatrix}$ in bases $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of spaces V, W, Z, respectively. Let $\alpha \in V$ have coordinates $1, -1, 3, -2$ with respect to A. Find the coordinates of $\varphi(\alpha)$ with respect to $\mathcal{B}$, the coordinates of $(\psi \circ \varphi)(\alpha)$ with respect to $\mathcal{C}$ and the matrix $M(\psi \circ \varphi)_{\mathcal{A}}^{\mathcal{C}}$.

$$M(\psi \circ \varphi)_{\mathcal{A}}^{\mathcal{C}} = M(\psi)_{\mathcal{B}}^{\mathcal{C}} \cdot M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 4 & 5 \\ 1 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 13 & 18 \\ 9 & 2 & 13 & 25 \\ 1 & 0 & 1 & 3 \end{bmatrix}.$$

Meanwhile, $((\psi \circ \varphi)(\alpha))_{\mathcal{C}} = M(\psi \circ \varphi)_{\mathcal{A}}^{\mathcal{C}} \cdot (\alpha)_{\mathcal{A}} = \begin{bmatrix} 7 & 3 & 13 & 18 \\ 9 & 2 & 13 & 25 \\ 1 & 0 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ -2 \end{bmatrix}$. So the coordinates are $7, -4, -2$.